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REGULARITY AND NUMERICAL SOLUTION OF EIGENVALUE  
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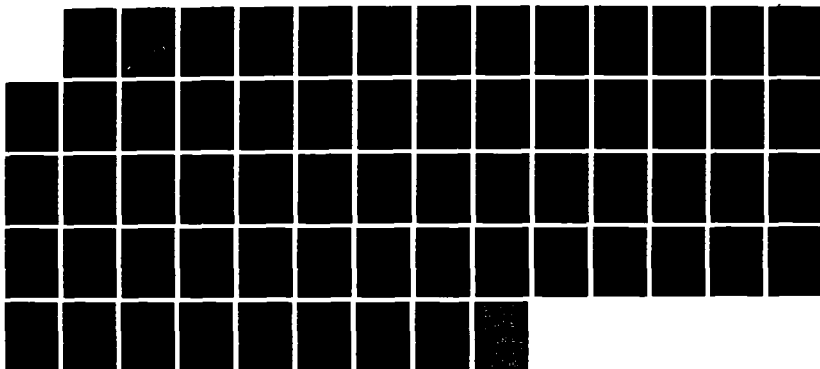
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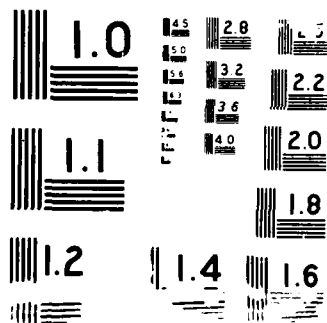
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REGULARITY AND NUMERICAL SOLUTION OF EIGENVALUE PROBLEMS  
WITH PIECEWISE ANALYTIC DATA

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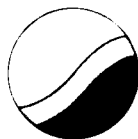
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### Abstract

In this paper we discuss<sup>26</sup> the regularity of the eigenfunctions of eigenvalue problems with piecewise analytic data and the approximation of the eigenvalues and eigenfunctions of such problems. A detailed and systematic numerical study of these approximations is presented, together with an analysis of the numerical results in light of the theoretical results. The specific aim is to assess the reliability of the theoretical results – which are of an asymptotic nature – as a guide to practical computations – which may take place in the preasymptotic phase – and to look for characteristic features of the numerical results which are not completely explained by known theoretical results.



# Regularity and Numerical Solution of Eigenvalue Problems with Piecewise Analytic Data

## 1. Introduction

The purpose of this paper is to discuss the regularity of the eigenfunctions of eigenvalue problems with piecewise analytic data and the approximation of the eigenpairs of such eigenvalue problems by the finite element method, and to present the results of a detailed and systematic numerical study of these approximations.

[1] contains an analysis of the regularity properties of the solutions of elliptic boundary value problems with piecewise analytic data. Specifically, it is shown that the solution of such a problem belongs to a countably normed space  $B^2_\beta(\Omega)$ . In [2] this fact is used to show that the  $h$ - $p$  version of the finite element method has an exponential rate of convergence. (For a survey of the basic results on the  $h$ - $p$  version of the finite element method and for relevant references we refer to [3] and [4].) In this paper we make a parallel study of the regularity and approximability of the eigenfunctions of eigenvalue problems with piecewise analytic data. We then discuss the implications of this approximability for the approximation of eigenvalues and eigenfunctions by the finite element method.

Section 2 contains background information, Section 3 describes the eigenvalue problems we treat, Section 4 presents a regularity result for eigenfunctions in terms of countably normed spaces, Section 5 surveys (abstract) results on eigenvalue and eigenfunction approximation by finite element methods, and Section 6 reports the results of a detailed and systematic numerical study of eigenvalue approximation by finite element methods and presents the conclusions of this study.

## 2. Notation and Preliminaries

Let  $\Omega \subset \mathbb{R}^2$  be a simply connected, bounded domain with boundary  $\Gamma = \partial\Omega$ , and assume that  $\Gamma = \bigcup_{i=1}^M \bar{\Gamma}_i$ , where  $\bar{\Gamma}_i$ , for  $i = 1, \dots, M$ , is an analytic simple arc connecting the vertices  $A_{i-1}$  and  $A_i$  ( $A_0 = A_M$ ).  $\Gamma_i$  will denote  $\bar{\Gamma}_i - \{A_{i-1}, A_i\}$ . The interior angle at  $A_i$  will be denoted by  $\omega_i$ . This notational scheme is shown in Figure 2.1.

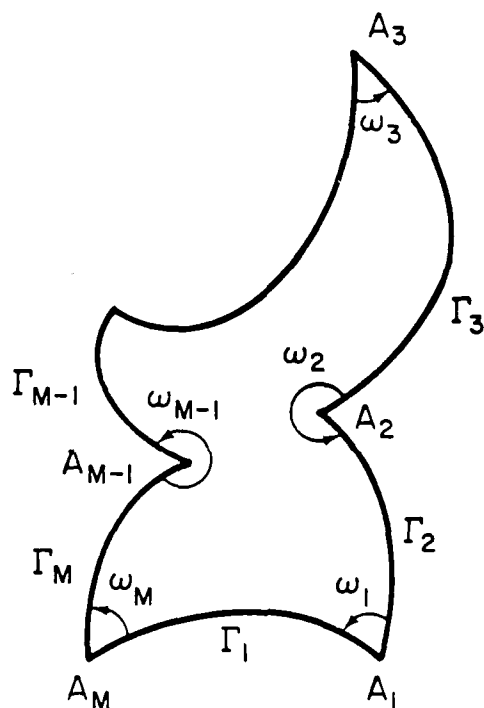


Figure 2.1. Notational Scheme for  $\Omega$  and  $\Gamma = \partial\Omega$ .

We will assume  $0 < \omega_i \leq 2\pi$ .  $\omega_i = 2\pi$  corresponds to a slit in  $\Omega$ . If  $\omega_i < 2\pi$  for all  $i$ , we will call  $\Omega$  a Lipschitz domain ( $\Gamma$  will be a Lipschitz curve in this case). If the arcs  $\Gamma_i$  are straight lines, we say  $\Omega$  is a straight polygon or simply a polygon; otherwise we will refer to  $\Omega$  as a curvilinear polygon. We let

$$\Gamma^0 = \bigcup_{i \in \mathcal{D}} \bar{\Gamma}_i,$$

where  $\mathcal{D}$  is a subset of  $\{1, \dots, M\}$ , and let

$$\Gamma^1 = \Gamma - \Gamma^0.$$

Then

$$\Gamma = \Gamma^0 \cup \Gamma^1.$$

$\Gamma^0$  will be referred to as the Dirichlet boundary and  $\Gamma^1$  as the Neumann boundary. We will often consider the special domains

$$S = S(\omega, \delta) = \{(r, \theta) : 0 < r < \delta, 0 < \theta < \omega\},$$

where  $(r, \theta)$  denotes the polar coordinates of  $x$ . In connection with  $S$  we introduce the notation

$$\tilde{\Gamma}_1(\omega, \delta) = \{(r, \theta) : 0 < r < \delta, \theta = 0\},$$

$$\tilde{\Gamma}_2(\omega, \delta) = \{(r, \theta) : 0 < r < \delta, \theta = \omega\}.$$

By  $H^m(\Omega)$ , for  $m \geq 0$  an integer, we denote the standard Sobolev space of functions  $u$  for which

$$(2.1) \quad \|u\|_{H^m(\Omega)}^2 = \sum_{|\alpha| \leq m} \|D^\alpha u\|_{L_2(\Omega)}^2 < \infty,$$

where  $D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}}$ , with  $\alpha = (\alpha_1, \alpha_2)$ ,  $\alpha_1, \alpha_2 \geq 0$  integers, and  $|\alpha| = \alpha_1 + \alpha_2$ . We note that  $\|u\|_{H^0(\Omega)} = \|u\|_{L_2(\Omega)}$ . Let

$$H_0^1(\Omega) = \{u : u \in H^1(\Omega), u = 0 \text{ on } \Gamma^0\};$$

on  $H_0^1(\Omega)$  we can use either  $\|u\|_{H^1(\Omega)}$  or  $\left[ \sum_{|\alpha|=1} \|D^\alpha u\|_{L_2(\Omega)}^2 \right]^{1/2}$ .

Let

$$r_j(x) = \text{dist}(x, A_j) = |x - A_j|,$$

$$r_j^*(x) = \min(1, r_j(x)),$$

$$\beta = (\beta_1, \dots, \beta_M), \quad \text{where } 0 < \beta_i < 1,$$

and

$k$  = an integer,

and define

$$\Phi_{\beta+k}(x) = \prod_{i=1}^M (r_i^*(x))^{\beta_i+k}.$$

If  $\Omega = S$ , we let  $\Phi_\beta = r^\beta$ . By  $H_\beta^{m,\ell}(\Omega)$ , with  $m$  and  $\ell$  integers satisfying  $0 \leq \ell \leq m$ , we denote the weighted Sobolev space characterized by the norm

$$(2.2) \quad \|u\|_{H_\beta^{m,\ell}(\Omega)}^2 = \begin{cases} \|u\|_{H^{\ell-1}(\Omega)}^2 + \sum_{k=\ell}^m \sum_{|\alpha|=k} \|\Phi_{\beta+k-\ell} D^\alpha u\|_{H^0(\Omega)}^2, & \text{if } \ell \geq 1 \\ \sum_{k=0}^m \sum_{|\alpha|=k} \|\Phi_{\beta+k} D^\alpha u\|_{H^0(\Omega)}^2, & \ell = 0. \end{cases}$$

$H_\beta^{0,0}(\Omega)$  will also be denoted by  $L_\beta(\Omega)$ .

We will also use the countably normed spaces

$$(2.3) \quad \mathcal{B}_\beta^\ell(\Omega) = \{u : u \in H_\beta^{\ell,\ell}(\Omega), \|\Phi_{\beta+k-\ell} D^\alpha u\|_{H^0(\Omega)} \leq C d^{k-\ell} (k-\ell)!,$$

for  $|\alpha| = k, k = \ell, \ell+1, \dots$ , where  $C \geq 1$  and

$d \geq 1$  are independent of  $k\}$

and

$$(2.4) \quad \mathcal{E}_\beta^2(\Omega) = \{u : u \in H_\beta^{2,2}(\Omega), |D^\alpha u(x)| \leq C d^k k! (\Phi_{\beta+k-1}(x))^{-1},$$

for  $|\alpha| = k, k = 1, 2, \dots$ , where  $C \geq 1$  and

$d \geq 1$  are independent of  $k\}$ .

In the case when  $\Omega = S(\omega, \delta)$  we let

$$\mathcal{D}^\alpha u = \frac{\partial^{|\alpha|} u}{\partial r^{\alpha_1} \partial \theta^{\alpha_2}} = u_{r^{\alpha_1} \theta^{\alpha_2}},$$

and then define the spaces  $\mathcal{H}_\beta^{m,\ell}(S)$ , with  $0 \leq \ell \leq m$ , and  $0 < \beta < 1$ , which are characterized by the norm

$$(2.5) \quad \|u\|_{\mathcal{H}_\beta^{m,\ell}(\Omega)}^2 = \begin{cases} \|u\|_{H^{\ell-1}(S)}^2 + \sum_{k=\ell}^m \sum_{|\alpha|=k} \|r^{\alpha_1-\ell+\beta} \mathcal{D}^\alpha u\|_{H^0(S)}^2, & \text{if } \ell \geq 1 \\ \sum_{k=0}^m \sum_{|\alpha|=k} \|r^{\alpha_1+\beta} \mathcal{D}^\alpha u\|_{H^0(S)}^2, & \ell = 0. \end{cases}$$

$\mathcal{H}_\beta^{0,0}(S)$  will also be denoted by  $\mathcal{L}_\beta(S)$ . Note that  $L_\beta(S) = \mathcal{L}_\beta(S)$ . Furthermore, we let  $\mathcal{H}^1(S)$  be the space characterized by the norm

$$\|u\|_{\mathcal{H}^1(S)}^2 = \|u\|_{H^0(S)}^2 + \sum_{|\alpha|=1} \|r^{\alpha_1-1} \mathcal{D}^\alpha u\|_{H^0(S)}^2.$$

Note that this is just  $\|u\|_{H^1(S)}^2$  expressed in polar coordinates.

We now state some lemmas we will use in the sequel.

**Lemma 2.1.** The spaces  $\mathcal{H}_\beta^{j,j}(S)$  and  $H_\beta^{j,j}(S)$  are equal for  $j = 0, 1, 2$ , i.e., the norms  $\|u\|_{\mathcal{H}_\beta^{j,j}(S)}$  and  $\|u\|_{H_\beta^{j,j}(S)}$  are equivalent for  $j = 0, 1, 2$ .

This result is obvious if  $j = 0, 1$ . For a proof in the case  $j = 2$ , see Lemma 1.1 of [1].

**Lemma 2.2.** Suppose  $0 \leq \ell \leq 2$  and  $k \geq \ell$ . Then

$$(2.6) \quad \|\Phi_{\beta+k-\ell} \mathcal{D}^\alpha u\|_{H^0(S)} \leq C d^{k-\ell} (k-\ell)! \quad \text{for any } \alpha$$

with  $|\alpha| = k$

if and only if

$$\begin{aligned}
(2.7) \quad & \left[ \int_S |D^{\alpha'} u|^2 r^{-2\alpha'} \Phi_{\beta+k'-\ell} r dr d\theta \right]^{1/2} \\
& \leq \tilde{C} d^{k'-\ell} (k'-\ell)! \quad \text{for all } \alpha' \\
& \text{with } \ell \leq |\alpha'| = k' \leq k.
\end{aligned}$$

For the proof of this result see Theorem 1.1 of [1].

Lemma 2.3. Let  $u \in H^1(S(\omega, \delta))$  satisfy

$$(2.8) \quad -\Delta u = f \quad \text{on } S(\omega, \delta)$$

and suppose

(a)

$$(2.8a) \quad u = 0 \quad \text{on } \tilde{\Gamma}_1 \cup \tilde{\Gamma}_2,$$

(b)

$$(2.8b) \quad \begin{cases} u = 0 & \text{on } \tilde{\Gamma}_1 \\ \frac{\partial u}{\partial n} = 0 & \text{on } \tilde{\Gamma}_2. \end{cases}$$

or

(c)

$$(2.8c) \quad \frac{\partial u}{\partial n} = 0 \quad \text{on } \tilde{\Gamma}_1 \cup \tilde{\Gamma}_2.$$

Then

$$(2.9) \quad \|u\|_{H_{\beta}^{2,2}(S(\omega, \delta/2))} \leq C \left[ \|f\|_{L_{\beta}(S(\omega, \delta))} + \|u\|_{H_{\beta}^{1,1}(S(\omega, \delta) - S(\omega, \delta/2))} \right],$$

for any  $\beta$  satisfying  $0 < \beta < 1$  and, in addition,  $1 - \pi/\omega < \beta$  in cases (a) and (c) and  $1 - \pi/2\omega < \beta$  in case (b).

For the proof of (2.9), see the proof of Theorem 2.1, specifically the proof of inequality (2.44), in [1].

Lemma 2.4.  $\mathcal{B}_\beta^2(\Omega) \subset \mathfrak{E}_\beta^2(\Omega) \subset \mathcal{B}_{\beta+\epsilon}^2(\Omega)$ , for any  $\epsilon > 0$ .

For the proof of this result see Theorems 2.2 and 2.3 of [5].

### 3. The Eigenvalue Problem

Let  $\Omega$  be a straight or curved polygon, and consider the selfadjoint eigenvalue problem

$$(3.1) \quad \begin{cases} (Lu)(x) = \lambda(Mu)(x), & x = (x_1, x_2) \in \Omega \\ (Bu)(x) = 0, & x \in \Gamma, \end{cases}$$

where

$$(3.2) \quad (Lu)(x) = - \sum_{i,j=1}^2 \frac{\partial}{\partial x_j} \left[ a_{ij}(x) \frac{\partial u}{\partial x_i}(x) \right] + c(x)u,$$

with  $a_{ij}(x) = a_{ji}(x)$  and  $c(x) \geq 0$  analytic functions on  $\bar{\Omega}$  satisfying

$$(3.3) \quad \sum_{i,j} a_{ij}(x) \xi_i \xi_j \geq \gamma_1 (\xi_1^2 + \xi_2^2) \quad \forall x \in \Omega \quad \text{and} \quad \forall (\xi_1, \xi_2), \quad \text{with} \quad \gamma_1 > 0,$$

where

$$(3.4) \quad (Mu)(x) = d(x)u(x),$$

with  $d(x)$  an analytic function on  $\bar{\Omega}$  satisfying

$$(3.5) \quad d(x) \geq \gamma_2 > 0,$$

and where

$$(3.6) \quad (Bu)(x) = \begin{cases} u(x), & x \in \Gamma^0 \\ \frac{\partial u}{\partial \nu}(x) = - \sum_{i,j} a_{ij} n_j \frac{\partial u}{\partial x_i}(x), & x \in \Gamma^1, \end{cases}$$

with  $n = n(x) = (n_1, n_2)$  denoting the exterior unit normal to  $\Gamma$  at  $x$ . If we define the bilinear forms

$$(3.7) \quad a(u, v) = \int_{\Omega} \left[ \sum_{i,j} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + cuv \right] dx$$



and

$$(3.8) \quad b(u, v) = \int_{\Omega} u v dx$$

on  $H_0^1(\Omega)$ , then (3.1) has the weak, or variational, formulation

$$(3.9) \quad \begin{cases} \text{Seek } \lambda \in \mathbb{R}, 0 \neq u \in H_0^1(\Omega) \text{ satisfying} \\ a(u, v) = \lambda b(u, v) \quad \forall v \in H_0^1(\Omega). \end{cases}$$

(3.9) has eigenvalues

$$0 < \lambda_1 \leq \lambda_2 \leq \dots \nearrow +\infty$$

and corresponding eigenfunctions

$$u_1, u_2, \dots$$

satisfying

$$a(u_i, u_j) = \delta_{ij}.$$

In Section 4 we prove the following theorem on the regularity of the eigenfunctions of (3.9).

Theorem 3.1. Suppose  $\Omega$  is a straight or Lipschitz curved polygon (as described in Section 2). Then for each  $j = 1, 2, \dots$ ,

$$u_j \in \mathcal{B}_{\beta}^2(\Omega),$$

where  $\beta = (\beta_1, \dots, \beta_M)$ , with  $\beta_i$  satisfying

$$0 \leq \beta_i^0 < \beta_i < 1,$$

where  $\beta_i^0$  depends on the values  $a_{mn}(A_i)$  and the angle  $\omega_i$ .

Remark 3.1. We have assumed  $\Omega$  is simply connected. Theorem 3.1 can easily be extended to cover multiply connected domains.

Remark 3.2. In Theorem 3.1 we have assumed that  $a_{ij}, c$ , and  $d$  are analytic on  $\bar{\Omega}$ . This condition can be weakened to the requirement that these functions are analytic on  $\bar{\Omega} - \bigcup_{i=1}^M A_i$ , provided we impose certain growth conditions on  $a_{ij}, c, d$ , and their derivatives.

#### 4. Regularity of the Eigenfunctions

In this section we will prove Theorem 3.1. We provide the complete details only for the case when  $Lu = -\Delta u$  and  $\Omega$  is a straight polygon with  $\Gamma^0 = \Gamma$  (so that the boundary conditions are of Dirichlet type). The proof in the general case will only be outlined since the arguments are similar to those used in [1].

Lemma 4.1. The eigenfunctions of (3.9) are analytic in  $\tilde{\Omega} = \bar{\Omega} - \bigcup_{i=1}^M A_i$ .

For the proof of this result see [6, Theorems 5.7.1 and 5.7.1'].

Proof of Theorem 3.1. Assume  $\Omega$  is a straight polygon, and suppose, without loss of generality, that  $A_i$  is located at the origin and  $\Gamma_{i+1}$  lies along the  $x_1$ -axis. Choose  $0 < \delta_i < 1$  so that  $S_{\delta_i} = S(\omega_i, \delta_i) \subset \Omega$  and  $\bar{S}_{\delta_i} \cap$

$\sum_{j \neq i, i+1} \bar{\Gamma}_j = \emptyset$ . See Figure 4.1.

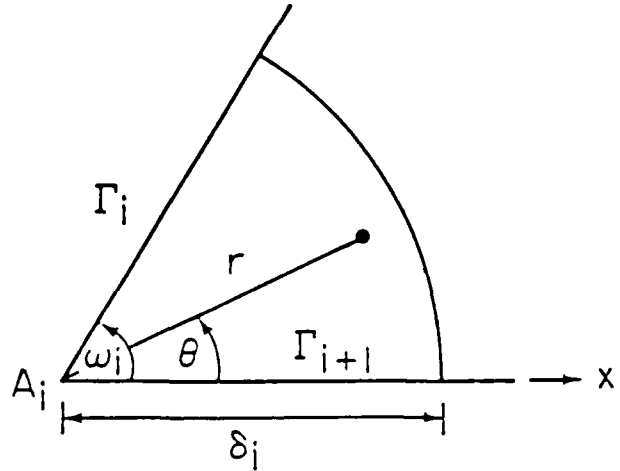


Figure 4.1. The Domain  $S_{\delta_i}$ .

Consider an eigenpair  $(\lambda, u)$  of (3.9) and suppose  $\|u\|_{H^1(\Omega)} = 1$ .

Localizing to the vertex  $A_i$  we let

$$(4.1) \quad v_k = r^k u_{r^k}, \quad k = 0, 1, 2, \dots$$

Then, using the identity

$$(r^k u_{rk}) = r^{k-1} (r^2 \Delta u)_{rk} = -\lambda r^{k-2} (r^2 u)_{rk}, \quad k \geq 0,$$

we have

$$(4.2) \quad \begin{cases} -\Delta v_k = \lambda r^{k-1} (r^2 u)_{rk} & \text{in } S_{\delta_1} \\ v_k = 0 & \text{on } \bar{\Gamma}_1 \cup \bar{\Gamma}_2. \end{cases}$$

By Lemma 4.1,  $u$  is analytic in  $\tilde{\Omega}$ . Thus there exist constants  $C_0$  and  $d_0$  such that

$$(4.3) \quad \|v_k\|_{H^1(S_{\delta_1} - S_{\delta_1/2})} \leq C_0 d_0^0 k!, \quad k = 0, 1, \dots$$

Let us now prove that there are numbers  $C$  and  $d$  such that

$$(4.4) \quad \|v_k\|_{H_{\beta}^{2,2}(S_{\delta_1/2})} \leq C d^k k!, \quad k = 0, 1, 2, \dots,$$

for any  $0 < \beta < 1$  satisfying  $1 - \pi/\omega_1 = \beta_1^0 < \beta$ . We will show, in fact, that (4.4) holds provided  $C = C_1(C_0 + \lambda + 1)$ ,  $d$  satisfies

$$\frac{d_0}{d} \geq 1,$$

$$\frac{3C_1 \lambda}{d} + \frac{C_1 \lambda}{d^2} \leq 1/2,$$

and

$$\frac{C_0 C_1 (\lambda + 1) d_0}{d} + \frac{2C_0 C_1 \lambda}{d} + \frac{C_0 C_1 \lambda}{d^2} \leq \frac{C}{2},$$

where  $C_0$  and  $C_1$  are the constants in (4.3) and (2.9), respectively, and  $\beta_1^0 < \beta$ . First consider  $k = 0$ .  $u = v_0 \in H^1(S_{\delta_1})$ ,  $u = v_0 = 0$  on  $\bar{\Gamma}_1 \cup \bar{\Gamma}_2$ ,

$-\Delta u = \lambda u$  on  $S_{\delta_1}$  (cf. (4.3)), and  $\|u\|_{H^1(\Omega)} = 1$ , so we can apply Lemma 2.3

to obtain

$$\begin{aligned}
\|v_0\|_{\mathcal{H}_\beta^{2,2}(S_{\delta_1/2})} &= \|u\|_{\mathcal{H}_\beta^{2,2}(S_{\delta_1/2})} \\
&\leq C_1 \left[ \lambda \|u\|_{\mathcal{L}_\beta(S_{\delta_1})} + \|u\|_{\mathcal{H}_\beta^{1,1}(S_{\delta_1} - S_{\delta_1/2})} \right] \\
&\leq C_1 \max(1, \delta_1^\beta) \left[ \lambda \|u\|_{H^0(S_{\delta_1})} + \|u\|_{H^1(S_{\delta_1} - S_{\delta_1/2})} \right] \\
&\leq C_1 (\lambda + 1) \|u\|_{H^1(\Omega)} \\
&\leq C_1 (\lambda + 1), \quad \text{provided } 0 < \beta < 1 \text{ satisfies} \\
&\quad 1 - \pi/\omega_1 = \beta_1^0 < \beta;
\end{aligned}$$

thus (4.4), with  $k = 0$ , holds for this choice of  $C$  and  $d$ .

Suppose (4.4) holds for  $k = 0, 1, \dots, \ell-1$ , where  $\ell \geq 1$ . We will show it holds for  $k = \ell$  and conclude by induction that it holds for all  $k$ . We begin by noting that  $v_\ell \in H^1(S_{\delta_1})$ . To see this it is sufficient to show that

$\|v_\ell\|_{H^1(S_{\delta_1/2})} < \infty$ . We easily see that

$$\begin{aligned}
\|v_\ell\|_{H^1(S_{\delta_1/2})} &= \|(1-\ell)v_{\ell-1} + r(v_{\ell-1})_r\|_{H^1(S_{\delta_1/2})} \\
&\leq C \left[ \|v_{\ell-1}\|_{H^1(S_{\delta_1/2})} + \|r(v_{\ell-1})_r\|_{H^1(S_{\delta_1/2})} \right] \\
&\leq C \left[ \|v_{\ell-1}\|_{H^1(S_{\delta_1/2})} + \|r(v_{\ell-1})_r\|_{H^0(S_{\delta_1/2})} \right. \\
&\quad \left. + \|[r(v_{\ell-1})_r]_r\|_{H^0(S_{\delta_1/2})} \right. \\
&\quad \left. + \|r^{-1}[r(v_{\ell-1})_r]_\theta\|_{H^0(S_{\delta_1/2})} \right]
\end{aligned}$$

$$\begin{aligned}
&\leq C \left[ \|v_{\ell-1}\|_{H^1(S_{\delta_1/2})} + \|r(v_{\ell-1})_r\|_{H^0(S_{\delta_1/2})} \right. \\
&\quad + \|r(v_{\ell-1})_{rr}\|_{H^0(S_{\delta_1/2})} \\
&\quad + \|(v_{\ell-1})_r\|_{H^0(S_{\delta_1/2})} \\
&\quad \left. + \|(v_{\ell-1})_{r\theta}\|_{H^0(S_{\delta_1/2})} \right].
\end{aligned}$$

From the induction hypothesis,

$$\begin{aligned}
\|v_{\ell-1}\|_{\mathcal{H}_\beta^{2,2}(S_{\delta_1/2})}^2 &= \|v_{\ell-1}\|_{H^1(S_{\delta_1/2})}^2 + \|(v_{\ell-1})_{rr}\|_{\mathcal{L}_\beta^2(S_{\delta_1/2})}^2 \\
&\quad + \|r^{-1}(v_{\ell-1})_{r\theta}\|_{\mathcal{L}_\beta^2(S_{\delta_1/2})}^2 \\
&\quad + \|r^{-2}(v_{\ell-1})_{\theta\theta}\|_{\mathcal{L}_\beta^2(S_{\delta_1/2})}^2 \\
&< \infty.
\end{aligned}$$

Since  $r \leq 1$  on  $S_{\delta_1/2}$  we see that  $\|v_\ell\|_{H^1(S_{\delta_1/2})} \leq C\|v_{\ell-1}\|_{\mathcal{H}_\beta^{2,2}(S_{\delta_1/2})}$  and

hence that  $\|v_\ell\|_{H^1(S_{\delta_1/2})} < \infty$ . Now, since  $v_\ell \in H^1(S_{\delta_1})$  and because of (4.3)

we can apply Lemma 2.3 to get, for  $0 < \beta < 1$  satisfying  $\beta_1^0 < \beta$ ,

$$\begin{aligned}
&\|v_\ell\|_{\mathcal{H}_\beta^{2,2}(S_{\delta_1/2})} \\
&\leq C_1 \left\{ \lambda \|r^{\ell-2}(r^2 u)\|_{\mathcal{L}_\beta^2(S_{\delta_1})} + \|v_\ell\|_{\mathcal{H}_\beta^{1,1}(S_{\delta_1} - S_{\delta_1/2})} \right\} \\
&\leq C_1 \max(1, \delta_1^\beta) \left\{ \lambda \|r^{\ell-2}(r^2 u)\|_{\mathcal{L}_\beta^0(S_{\delta_1})} + \|v_\ell\|_{\mathcal{H}_\beta^1(S_{\delta_1} - S_{\delta_1/2})} \right\}
\end{aligned}$$

$$= C_1 \left\{ \lambda \| r^{\ell-2}(r^2 u)_{r^\ell} \|_{H^0(S_{\delta_1})} + \| v_\ell \|_{H^1(S_{\delta_1}-S_{\delta_1/2})} \right\}.$$

Then we note that

$$(4.6) \quad r^{\ell-2}(r^2 u)_{r^\ell} = v_\ell + 2\ell v_{\ell-1} + \ell(\ell-1)v_{\ell-2}.$$

$$(4.7) \quad \begin{aligned} & \| r^{\ell-2}(r^2 u)_{r^\ell} \|_{H^0(S_{\delta_1})} \\ & \leq \| r^{\ell-2}(r^2 u)_{r^\ell} \|_{H^0(S_{\delta_1}-S_{\delta_1/2})} + \| r^{\ell-2}(r^2 u)_{r^\ell} \|_{H^0(S_{\delta_1/2})} \end{aligned}$$

$$(4.8) \quad \begin{aligned} \| v_\ell \|_{H^0(S_{\delta_1/2})} &= \| (1-\ell)v_{\ell-1} + r(v_{\ell-1})_{r^\ell} \|_{H^0(S_{\delta_1/2})} \\ &\leq (\ell-1) \| v_{\ell-1} \|_{H^0(S_{\delta_1/2})} + \| r(v_{\ell-1})_{r^\ell} \|_{H^0(S_{\delta_1/2})} \\ &\leq \ell \| v_{\ell-1} \|_{H^1(S_{\delta_1/2})} \\ &\leq \ell \| v_{\ell-1} \|_{H^{2,2}_\beta(S_{\delta_1/2})}. \end{aligned}$$

Now, using (4.3), (4.6), (4.7), (4.8), and the induction hypothesis, we obtain

$$\begin{aligned} & \| r^{\ell-2}(r^2 u)_{r^\ell} \|_{H^0(S_{\delta_1})} \\ & \leq \| r^{\ell-2}(r^2 u)_{r^\ell} \|_{H^0(S_{\delta_1}-S_{\delta_1/2})} + \| r^{\ell-2}(r^2 u)_{r^\ell} \|_{H^0(S_{\delta_1/2})} \\ & \leq \| v_\ell \|_{H^0(S_{\delta_1}-S_{\delta_1/2})} + 2\ell \| v_{\ell-1} \|_{H^0(S_{\delta_1}-S_{\delta_1/2})} \\ & \quad + \ell(\ell-1) \| v_{\ell-2} \|_{H^0(S_{\delta_1}-S_{\delta_1/2})} \end{aligned}$$

$$\begin{aligned}
& + \|v_\ell\|_{H^0(S_{\delta_1/2})} + 2\ell \|v_{\ell-1}\|_{H^0(S_{\delta_1/2})} \\
& + \ell(\ell-1) \|v_{\ell-2}\|_{H^0(S_{\delta_1/2})} \\
& \leq 3\ell \|v_{\ell-1}\|_{\mathcal{H}_\beta^{2,2}(S_{\delta_1/2})} + \ell(\ell-1) \|v_{\ell-2}\|_{\mathcal{H}_\beta^{2,2}(S_{\delta_1/2})} \\
& + \|v_\ell\|_{\mathcal{H}^1(S_{\delta_1}-S_{\delta_1/2})} + 2\ell \|v_{\ell-1}\|_{\mathcal{H}^1(S_{\delta_1}-S_{\delta_1/2})} \\
& + \ell(\ell-1) \|v_{\ell-2}\|_{\mathcal{H}^1(S_{\delta_1}-S_{\delta_1/2})} \\
& \leq \ell! [C(3d^{\ell-1} + d^{\ell-2}) + C_0(d_0^\ell + 2d_0^{\ell-1} + d_0^{\ell-2})].
\end{aligned}$$

Note that if  $\ell = 1$ , the terms  $d^{\ell-2}$  and  $d_0^{\ell-2}$  are not present. Combining (4.3), (4.5), and (4.9) we get

$$\begin{aligned}
(4.10) \quad \|v_\ell\|_{\mathcal{H}_\beta^{2,2}(S_{\delta_1/2})} & \leq \ell! \{CC_1\lambda(3d^{\ell-1} + d^{\ell-2}) \\
& + C_0C_1\lambda(d_0^\ell + 2d_0^{\ell-1} + d_0^{\ell-2}) + C_0C_1d_0^\ell\} \\
& = \ell! d^\ell \left\{ C \left[ \frac{3C_1\lambda}{d} + \frac{C_1\lambda}{d^2} \right] \right. \\
& + C_0C_1(\lambda+1) \left[ \frac{d_0}{d} \right]^\ell + 2C_0C_1\lambda \left[ \frac{d_0}{d} \right]^{\ell-1} \frac{1}{d} \\
& \left. + C_0C_1\lambda \left[ \frac{d_0}{d} \right]^{\ell-2} \frac{1}{d^2} \right\} \\
& \leq \ell! d^\ell \left\{ C \left[ \frac{3C_1\lambda}{d} + \frac{C_1\lambda}{d^2} \right] \right. \\
& \left. + \frac{C_0C_1(\lambda+1)d_0}{d} + \frac{2C_0C_1\lambda}{d} + \frac{C_0C_2\lambda}{d^2} \right\}
\end{aligned}$$



$$\leq \ell! d^{\ell} \left[ \frac{C}{2} + \frac{C}{2} \right]$$

$$= C \ell! d^{\ell};$$

thus (4.4), for  $k = \ell$ , holds with  $C$  and  $d$  as selected above. This completes the proof of (4.4) for all  $k \geq 0$ .

Next we show that there are constants  $\bar{C}$  and  $\bar{d}$  such that

$$(4.11) \quad \|r^{\alpha_1-2} \mathcal{D}^{\alpha} u\|_{\mathcal{L}_{\beta}(S_{\delta_1/2})} \leq \bar{C} \bar{d}^k k!,$$

for all  $\alpha$  with

$$|\alpha| = k+2 \geq 2 \quad \text{and} \quad \alpha_2 \leq 2,$$

provided  $\beta_1^0 < \beta$ . We, in fact, show that (4.11) holds with  $\bar{C} = 3C$  and  $\bar{d} = \max(6, d)$ , where  $C$  and  $d$  are the constants in (4.4). Let  $k \geq 0$ . Then there are three choices for  $\alpha$  with  $|\alpha| = k+2$  and  $\alpha_2 \leq 2$ , namely  $\alpha = (\alpha_1, \alpha_2)$  with  $\alpha_1 = k$  and  $\alpha_2 = 2$ ,  $\alpha_1 = k+1$  and  $\alpha_2 = 1$ , and  $\alpha_1 = k+2$  and  $\alpha_2 = 0$ . Thus (4.11) is equivalent to the following three inequalities:

$$(4.12a) \quad \|r^{k-2} u_{r^{k+2}\theta}\|_{\mathcal{L}_{\beta}(S_{\delta_1/2})} \leq \bar{C} \bar{d}^k k!,$$

$$(4.12b) \quad \|r^{k-1} u_{r^{k+1}\theta}\|_{\mathcal{L}_{\beta}(S_{\delta_1/2})} \leq \bar{C} \bar{d}^k k!,$$

$$(4.12c) \quad \|r^k u_{r^{k+2}}\|_{\mathcal{L}_{\beta}(S_{\delta_1/2})} \leq \bar{C} \bar{d}^k k!, \quad k \geq 0.$$

Recall that (4.4) states that

$$(4.13) \quad \|v_k\|_{\mathcal{H}_{\beta}^{2,2}(S_{\delta_1/2})}^2 = \|r^k u_{r^k}\|_{H^1(S_{\delta_1/2})}^2 + \|r^{\beta}(r^k u_{r^k})_{rr}\|_{H^0(S_{\delta_1/2})}^2 \\ + \|r^{\beta-1}(r^k u_{r^k})_{r\theta}\|_{H^0(S_{\delta_1/2})}^2$$

$$\begin{aligned}
& + \|r^{\beta-2}(r^k u_{r^k})_{\theta\theta}\|_{H^0(S_{\delta_1/2})}^2 \\
& \leq C d^k k!.
\end{aligned}$$

(4.12a) follows immediately from (4.13). Next, consider (4.12b). For  $k = 0$ , it follows directly from (4.13). Suppose now that (4.12b) holds for  $0, 1, \dots, k-1$ . Then, using (4.13), we have

$$\begin{aligned}
\|r^{k-1} u_{r^{k+1}\theta}\|_{\mathcal{L}_\beta(S_{\delta_1/2})} &= \|r^{\beta-1}(r^k u_{r^k})_{r\theta} - k r^{\beta+k-2} u_{r^k\theta}\|_{H^0(S_{\delta_1/2})} \\
&\leq C d^k k! + \bar{C} \bar{d}^{k-1} k! \\
&= C k! (d^k + 3 \bar{d}^{k-1}) \\
&\leq 2 C k! \bar{d}^k \\
&\leq \bar{C} \bar{d}^k k!,
\end{aligned}$$

which shows, by induction, that (4.12b) holds for all  $k$ . Finally consider (4.12c). For  $k = 0$ , this follows directly from (4.13). Suppose now that (4.12c) holds for  $0, 1, \dots, k-1$ . Then, again using (4.13), we have

$$\begin{aligned}
\|r^k u_{r^{k+2}}\|_{\mathcal{L}_\beta(S_{\delta_1/2})} &= \|r^\beta (r^k u_{r^k})_{rr} \\
&\quad - 2k r^{\beta+k-1} u_{r^{k+1}} - k(k-1) r^{\beta+k-2} u_{r^k}\|_{H^0(S_{\delta_1/2})} \\
&\leq C d^k k! + 2 \bar{C} \bar{d}^{k-1} k! + \bar{C} \bar{d}^{k-2} k! \\
&= C k! (d^k + 6 \bar{d}^{k-1} + 3 \bar{d}^{k-2}) \\
&\leq (3C) k! \bar{d}^k \\
&= \bar{C} \bar{d}^k k!,
\end{aligned}$$

which shows that (4.12c) holds for all  $k$ . This completes the proof of (4.11).

Using the identity

$$\begin{aligned}\Delta(r^{\alpha_1} u_{r^{\alpha_1} \theta^{\alpha_2-2}}) &= r^{\alpha_1-2} (r^2 \Delta u_{\theta^{\alpha_2-2}})_{r^{\alpha_1}} \\ &= -\lambda r^{\alpha_1-2} (r^2 u_{\theta^{\alpha_2-2}})_{r^{\alpha_1}},\end{aligned}$$

which is valid for  $\alpha_2 \geq 2$ , we get

$$\begin{aligned}(4.14) \quad r^{\alpha_1-2} u_{r^{\alpha_1} \theta^{\alpha_2}} &= -(r^{\alpha_1} u_{r^{\alpha_1} \theta^{\alpha_2-2}})_{r^2} \\ &\quad - r^{-1} (r^{\alpha_1} u_{r^{\alpha_1} \theta^{\alpha_2-2}})_r \\ &\quad - \lambda r^{\alpha_1-2} (r^2 u_{\theta^{\alpha_2-2}})_{r^{\alpha_1}}.\end{aligned}$$

Note that the number of  $\theta$ -differentiations on the right side of (4.14) is two less than on the left. Thus repeated application of (4.14) reduces the number of  $\theta$ -differentiations to  $\leq 2$ , so that (4.11) can be applied. In this way one shows that there exists  $\bar{C}$  and  $\bar{d}$  so that

$$(4.15) \quad \|r^{\alpha_1-2} D^\alpha u\|_{\mathcal{B}_{\beta}(S_{\delta_1/2})} \leq \bar{C} \bar{d}^k k!, \quad \text{for } |\alpha| = k+2 \geq 2,$$

provided  $\beta_1^0 < \beta$ .

Now  $u \in \mathcal{B}_{\beta}^2(S_{\delta_1/2})$  if and only if  $u \in H_{\beta}^{2,2}(S_{\delta_1/2})$  and

$$\begin{aligned}(4.16) \quad \|r^{\beta+k-2} D^\alpha u\|_{H^0(S_{\delta_1/2})} &\leq C d^{k-2} (k-2)!, \\ |\alpha| &= k, \quad k = 2, 3, \dots,\end{aligned}$$

for some  $C$  and  $d$  independent of  $k$ . (4.4), with  $k = 0$ , together with Lemma 2.1, shows that  $u \in H_{\beta}^{2,2}(S_{\delta_1/2})$ . It thus remains to prove (4.16). Let  $k \geq 2$  be fixed. By Lemma 2.2,

$$\|r^{\beta+k-2} D^\alpha u\|_{H^0(S_{\delta_1/2})} \leq C d^{k-2} (k-2)!, \text{ for } |\alpha| = k$$

if and only if

$$(4.17) \quad \|r^{k'-2-\alpha'} D^{\alpha'} u\|_{\mathcal{L}_\beta(S_{\delta_1/2})} \leq \tilde{C} d^{k'-2} (k'-2)!,$$

for  $2 \leq |\alpha'| = k' \leq k$ .

But from (4.15) we have

$$\|r^{k'-2-\alpha'} D^{\alpha'} u\|_{\mathcal{L}_\beta(S_{\delta_1/2})} \leq \bar{C} d^{k'-2} (k'-2)!,$$

for  $2 \leq |\alpha'| = k' \leq k$ ,

which implies (4.17). Thus (4.16) holds for some  $C$  and  $d$ , and we conclude that  $u \in \mathcal{B}_\beta^2(S_{\delta_1/2})$  for any  $0 < \beta < 1$  satisfying  $1 - \pi/\omega_1 = \beta_1^0 < \beta$ .

In this way we prove that  $u \in \mathcal{B}_{\beta_1}^2(S_{\delta_1/2})$  for  $\beta_1^0 < \beta_1$  in the neighborhood of each vertex  $A_1$ . Combining this result with Lemma 4.1 we arrive at the proof of Theorem 3.1 in the case that  $L = -\Delta$  and  $\Omega$  is a straight polygon. If  $L$  is the general operator in (3.2) and  $\Omega$  is a straight polygon, the result is obtained using the techniques in Section 3.2 of [5]. Finally, for the general case in which  $\Omega$  is a Lipschitz curved polygon we employ the arguments used in the proof of Theorem 3.4 of [5] to get the desired result. More specifically, by that technique we show that  $u \in \mathcal{C}_\beta^2(\Omega)$ . Then, using Lemma 2.4, we get Theorem 3.1 in its full generality.

Remark 4.1. A careful analysis of the proof of Theorem 3.1 allows one to assess the dependence on  $\lambda$  of the values at the constants  $C$  and  $d$  (cf. (4.16)), which are related to the smoothness of the eigenfunctions. One can see, for example, that  $d = \hat{d}\lambda + \hat{k}$ , where  $\hat{d} \geq 1$  and  $\hat{k}$  are independent of  $\lambda$ . This suggests that the higher eigenfunctions are less smooth than the

lower ones. Although this is, in general, correct, we shall see in Section 6 that there are important exceptions. In the example treated there we will see that of the first three eigenfunctions, the first is the roughest while the third is the smoothest.

## 5. Basic Results on Eigenvalue and Eigenfunction Approximation

As we have seen in Section 3, the eigenvalue problems we are considering have the form

$$(5.1) \quad \begin{cases} \text{Seek } \lambda \in \mathbb{R}, 0 \neq u \in H_0^1(\Omega) \text{ satisfying} \\ a(u, v) = \lambda b(u, v) \quad \forall v \in H_0^1(\Omega), \end{cases}$$

where the bilinear forms  $a$  and  $b$  are defined in (3.7) and (3.8).

We are interested in approximating the eigenpairs  $(\lambda, u)$  of (5.1), and toward this end we select a family of finite dimensional subspaces  $\hat{S}(p, k)$  of  $H_0^1(\Omega)$ , indexed by the parameters  $p = 1, 2, \dots$  and  $k = 0, 1, \dots$ , satisfying

$$(5.2) \quad \inf_{\chi \in \hat{S}(p, k)} \|u - \chi\|_{H_0^1(\Omega)} \rightarrow 0 \text{ as } p \rightarrow \infty, \text{ uniformly in } k, \\ \text{for each } u \in H_0^1(\Omega)$$

(the reason for employing two parameters will be made clear later). Then we consider the finite dimensional eigenvalue problem

$$(5.3) \quad \begin{cases} \text{Seek } \lambda(p, k) \in \mathbb{R}, 0 \neq u(p, k) \in \hat{S}(p, k) \text{ satisfying} \\ a(u(p, k), v) = \lambda(p, k) b(u(p, k), v) \quad \forall v \in \hat{S}(p, k). \end{cases}$$

(5.3) has a sequence of eigenvalues

$$0 < \lambda_1(p, k) \leq \dots \leq \lambda_N(p, k)$$

and corresponding eigenfunctions

$$u_1(p, k), \dots, u_N(p, k)$$

satisfying

$$a(u_i(p, k), u_j(p, k)) = \delta_{ij},$$

where  $N = N(p, k) = \dim \hat{S}(p, k)$ . The eigenpairs  $(\lambda_i(p, k), u_i(p, k))$  of (5.2) are viewed as approximations to the eigenpairs  $(\lambda_i, u_i)$  of (5.1). It is

well-known (see, e.g., [7]) that

$$(5.4) \quad \lambda_i \leq \lambda_i(p, k), \quad \forall p, k, \quad i = 1, \dots, N,$$

and, as a consequence of (5.2) (see, e.g., [7]), that

$$\lim_{p \rightarrow \infty} \lambda_i(p, k) = \lambda_i, \quad \text{uniformly in } k, \quad \text{for each } i.$$

We now present the error estimates that describe the quality of the approximation  $(\lambda_i(p, k), u_i(p, k))$ . For the sake of simplicity, we state these results in the case in which the eigenvalues  $\lambda_i$  of (5.1) are simple. For a complete treatment of the general case we refer the reader to [7, 8, 9, 10, 11].

Let

$$\|u\|_E = \sqrt{a(u, u)}$$

and

$$\|u\|_b = \sqrt{b(u, u)}.$$

Then we let

$$(5.5) \quad \epsilon_i(p, k) = \inf_{\chi \in \hat{S}(p, k)} \|u_i - \chi\|_E,$$

$$(5.6) \quad \nu(p, k) = \sup_{\substack{f \in H_0^1(\Omega) \\ \|f\|_E = 1}} \inf_{\chi \in \hat{S}(p, k)} \|Tf - \chi\|_E,$$

and

$$(5.7) \quad \eta(p, k) = \sup_{\substack{f \in L_2(\Omega) \\ \|f\|_0 = 1}} \inf_{\chi \in \hat{S}(p, k)} \|Tf - \chi\|_E,$$

where  $T : L_2(\Omega) \rightarrow H_0^1(\Omega)$  is the solution operator corresponding to the differential operator  $L$  introduced in (3.1), i.e.,  $T$  is defined by

$$(5.8) \quad \begin{cases} Tf \in H_0^1(\Omega) \\ a(Tf, v) = b(f, v) \quad \forall v \in H_0^1(\Omega). \end{cases}$$

The error estimates are given in terms of the quantities  $\epsilon_i(p, k)$ ,  $\nu(p, k)$ , and  $\eta(p, k)$ . We note that all of these quantities approach zero as  $p \rightarrow \infty$ , uniformly in  $k$ .

Theorem 5.1. Suppose  $\lambda_i$  is simple. Then there is a constant  $d_i$  such that

$$(5.9) \quad 1 - d_i \eta^2(p, k) \leq \frac{(\lambda_i(p, k) - \lambda_i)/\lambda_i}{\epsilon_i^2(p, k)} \leq 1 + d_i \nu(p, k),$$

for all  $p$  and  $k$ ,

and

$$(5.10) \quad 1 \leq \frac{\|u_i(p, k) - u_i\|_E / \|u_i\|_E}{\epsilon_i(p, k)} \leq 1 + d_i \nu(p, k),$$

for all  $p$  and  $k$ .

This result shows that for  $k$  fixed and  $p$  large, the eigenvalue error

$$(\lambda_i(p, k) - \lambda_i)/\lambda_i$$

is nearly equal to

$$\epsilon_i^2(p, k),$$

and the eigenfunction error

$$\|u_i(p, k) - u_i\|_E / \|u_i\|_E$$

is nearly equal to

$$\epsilon_i(p, k).$$

One of the goals of this paper is to make a computational study of these error assessments in the practical range of the parameters  $p$  and  $k$ .

By the usual duality argument we have



$$(5.11) \quad \|u_i - E(p, k)u_i\|_b \leq C\eta(p, k)\|u_i - E(p, k)u_i\|_E,$$

where  $E(p, k)u$  is the  $a$ -orthogonal projection of  $u_i$  onto  $\hat{S}(p, k)$ , showing that  $\|u_i - E(p, k)u_i\|_b = o(\|u_i - E(p, k)u_i\|_E)$ . A parallel result holds for the eigenfunction error  $u_i - u_i(p, k)$ . We state this in

Theorem 5.2. Suppose  $\lambda_i$  is simple. Then there is a constant  $C_i$  such that

$$(5.12) \quad \|u_i - u_i(p, k)\|_b \leq C_i\eta(p, k)\|u_i - u_i(p, k)\|_E.$$

This result is proved in [11].

## 6. Numerical Results and Their Analysis

In this section we consider a model problem, present numerical results for it, and analyze these numerical results in light of the theoretical results outlined in Section 5. We will consider the behavior of both the  $p$  and  $h$ - $p$  versions of the finite element method. The specific aim is to assess the reliability of the theoretical results – which are of an asymptotic nature – as a guide to practical computations – which may take place in the pre-asymptotic phase – and to look for characteristic features of the numerical results which are not completely explained by known theoretical results.

### The Model Problem

Let  $\Omega$  be the L-shaped domain shown in Figure 6.1 and consider the eigenvalue problem

$$(6.1) \quad \begin{cases} -\Delta u = \lambda u, & x \in \Omega \\ u = 0, & x \in \Gamma = \partial\Omega. \end{cases}$$

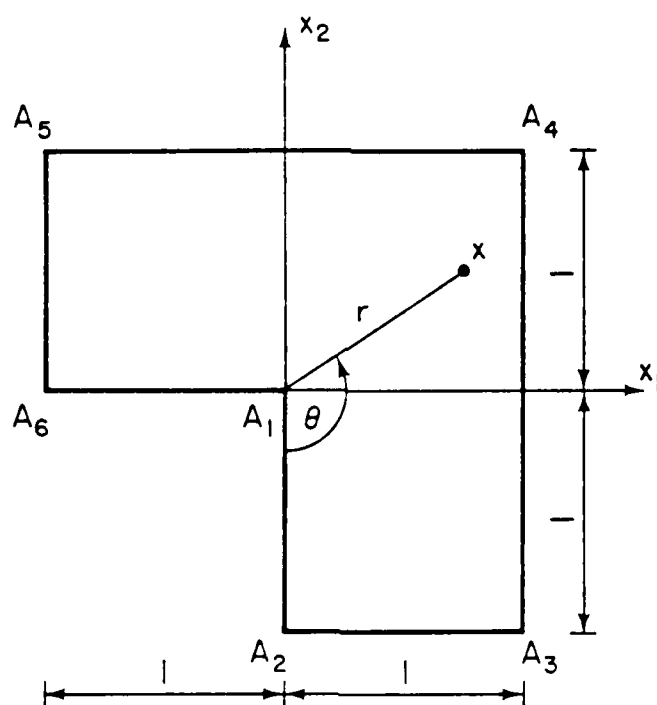


Figure 6.1. The Domain  $\Omega$

$\Omega$  is a Lipschitz straight polygon, as described in Section 2, and (6.1) is an eigenvalue problem of the type described in Section 3. We are taking  $\Gamma^0 = \Gamma$ . The bilinear forms  $a$  and  $b$  are given by

$$(6.2) \quad a(u, v) = \int_{\Omega} \left( \frac{\partial u}{\partial x_1} \frac{\partial v}{\partial x_2} + \frac{\partial u}{\partial x_2} \frac{\partial v}{\partial x_1} \right) dx$$

and

$$(6.3) \quad b(u, v) = \int_{\Omega} uv \, dx,$$

and the norms  $\|u\|_E$  and  $\|u\|_b$  are given by

$$(6.4) \quad \|u\|_E = \left( \int_{\Omega} \left[ \left( \frac{\partial u}{\partial x_1} \right)^2 + \left( \frac{\partial u}{\partial x_2} \right)^2 \right] dx \right)^{1/2}$$

and

$$(6.5) \quad \|u\|_b = \left( \int_{\Omega} u^2 \, dx \right)^{1/2}.$$

Thus  $\|u\|_E$  is the usual energy norm in  $H_0^1(\Omega)$  and  $\|u\|_b$  the usual  $L_2$ -norm. Let

$$0 < \lambda_1 \leq \lambda_2 \leq \dots \nearrow \infty$$

and

$$u_1, u_2, \dots$$

be the eigenvalues and eigenvectors of (6.1) and suppose

$$a(u_i, u_j) = \delta_{ij}.$$

The model problem (6.1) was selected as a typical example of a problem on a domain with nonsmooth, but piecewise analytic, boundary. It will also be of

interest in that it has some symmetries. (6.1) is a reasonable representation of a large class of engineering problems.

### The Regularity of the Eigenfunctions

On the basis of Theorem 3.1 we can make an assessment of the regularity of the eigenfunctions of (6.1). It is easy to see that the eigenfunctions of (6.1) are analytic at the vertices  $A_j$ , where  $\omega_j = \pi/2$ ,  $j = 2, \dots, 6$ , but that at  $A_1$ , where  $\omega_1 = \frac{3}{2}\pi$ , the eigenfunctions are singular. From Theorem 3.1, more specifically from the results of Section 4, we see that

$$(6.6) \quad u_1 = \mathcal{B}_\beta^2(\Omega),$$

where  $\beta = (\beta_1, \dots, \beta_6)$  with  $0 < \beta_2, \dots, \beta_6 < 1$  arbitrary (in fact, we can take  $\beta_2, \dots, \beta_6 = 0$ ) and  $1 - \frac{\pi}{\omega_1} = \frac{1}{3} < \beta_1 < 1$ .

We can also use the more standard regularity theory for problems in domains with corners (see, e.g., [12]) to analyze the singularities of the eigenfunctions. We have, for example,

$$(6.7) \quad u_1 = (C_1 r^{2/3} \sin 2/3 \theta) \psi + \nu_1,$$

where  $\psi$  is  $C^\infty$  cut-off function based at the point  $A_1$ ,  $\nu_1 \in H^2(\Omega)$ ,  $(r, \theta)$  are the polar coordinates depicted in Figure 6.1, and  $C_1 \neq 0$ . Obviously  $u_1 \in H^\alpha(\Omega)$ , with  $\alpha < 1 + 2/3$ , but  $u_1 \notin H^{5/3}(\Omega)$ . Nevertheless  $u_1 \in \mathcal{B}_{2,\omega}^{5/3}(\Omega)$ . (Here  $H^\alpha(\Omega)$  is the fractional Sobolev space and  $\mathcal{B}_{2,\omega}^\alpha(\Omega)$  is the Besov space; see [13]). Not all the eigenfunctions have the form (6.7). In fact, symmetry considerations show that  $u_2$  is antisymmetric with respect to the line  $x = -y$  and hence

$$(6.8) \quad u_2 = (C_2 r^{4/3} \sin 4/3 \theta) \psi + \nu_2,$$

where  $\nu_2 \in H^3(\Omega)$  and  $C_2 \neq 0$ , so that  $u_2 \in H^\alpha(\Omega)$ , for any  $\alpha < 1 + 4/3$ ,

and  $u_2 \in B_{2,\infty}^{1+4/3}(\Omega)$ . Furthermore,  $u_3$  is antisymmetric with respect to both axes and thus

$$(6.9) \quad u_3 = \left(\frac{2}{3}\right)^{1/2} \frac{1}{\pi} \sin \pi x_1 \sin \pi x_2;$$

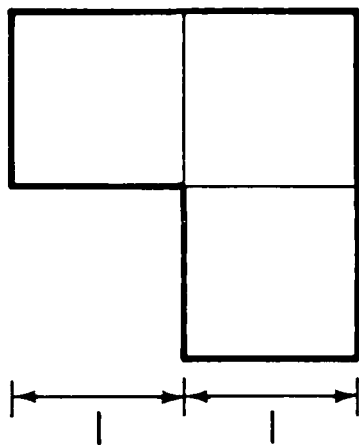
$u_3$  is therefore analytic on  $\bar{\Omega}$ .

We see here a typical phenomena for eigenvalue problems arising in engineering, namely, because of the presence of various symmetries, higher eigenfunctions may be smoother and hence lead to more accurate numerical approximations than the lower ones. See also Remark 4.1.

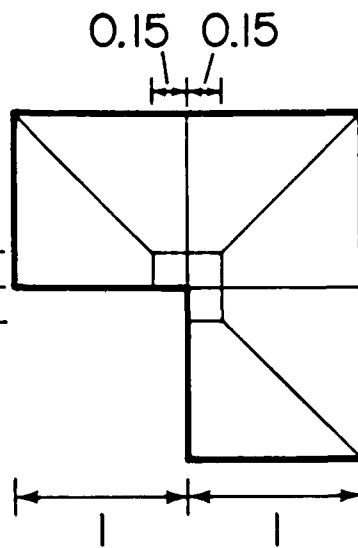
### The Finite Element Spaces

We consider the meshes shown in Figure 6.2. These are typical meshes for the  $p$  and  $h$ - $p$  versions of the finite element method for the approximation of (6.1) since the leading singularities of the eigenfunctions are located at vertex  $A_1$  (for more, see [2] and [14]). The meshes are characterized by the number  $k$  of layers.

$k=0$

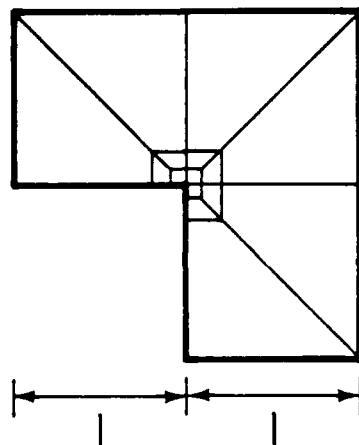


$0.15$

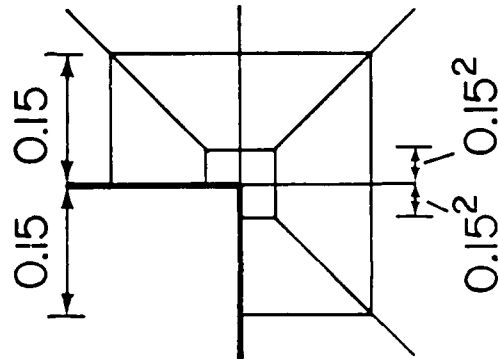


$k=1$

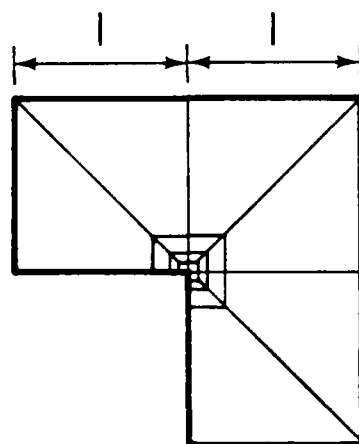
$k=2$



DETAIL



$k=3$



DETAIL

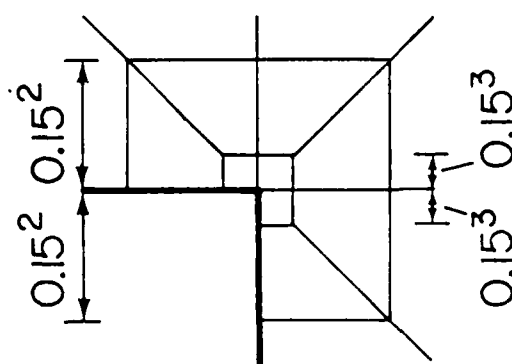


Figure 6.2. Meshes for the Finite Element Spaces.

By  $S(p,k)$  we denote the finite element space consisting in polynomials of degree  $p$  on a mesh with  $k$  layers. The functions on the trapezoidal elements are defined, as usual, as pull-back polynomials of degree  $p$  based on the standard bilinear map. The elements are of serendipity type  $Q'$  (see [15]). The number of degrees of freedom of one (unconstrained) quadrilateral element is

$$\begin{cases} 4p, & \text{for } p < 4 \\ 4p + \frac{(p-2)(p-3)}{2}, & \text{for } p \geq 4. \end{cases}$$

By  $\hat{S}(p,k)$  we denote the finite element space constrained by zero on  $\Gamma$ , i.e.,  $\hat{S}(p,k) = S(p,k) \cap H_0^1(\Omega)$ . The number of degrees of freedom is defined to be the dimension of  $\hat{S}(p,k)$  and will be denoted by  $N(p,k)$ . For large  $p$  and  $k$  we obviously have

$$N(p,k) \approx (k+1)p^2.$$

For small  $p$  and  $k$  this asymptotic formula is, of course, not very accurate. Table 6.1 gives  $N(p,k)$  for various combinations of  $p$  and  $k$  under consideration. The ratio  $N(p,k)/(k+1)p^2$  ranges from .44 to 3.75 over our range of  $p$  and  $k$ .

$\begin{smallmatrix} p \\ k \end{smallmatrix}$	1	2	3	4	5	6	7	8
0	0	2	4	9	17	28	42	59
1	5	18	31	53	84	124	173	231
2	10	34	58	97	151	220	304	403
3	15	50	85	141	218	316	435	575

Table 6.1. Number of Degrees of Freedom

We recall that in the  $p$  version, accuracy is achieved by letting  $p \rightarrow \infty$  while  $k$  is held fixed (i.e., the mesh is held fixed). With the  $h$ - $p$  version, we simultaneously increase  $p$  and  $k$ . In the classical  $h$  version, accuracy is achieved by refining the mesh while keeping  $p$  fixed. We will concentrate on the  $p$  and  $h$ - $p$  versions. We will not study the  $h$  version nor assess the question of how the performance of elements of type  $Q'$  (serendipity type), which we are using, compares with that of the full tensor product elements (of type  $Q$ ) or with that of triangular elements. Nevertheless, we will consider the effect of distortion of elements for the  $p$  and  $h$ - $p$  versions. For  $k = 0$ , none of the elements are distorted, while for  $k > 0$  many elements are trapezoidal, and hence, distorted.

#### Approximation Properties of the Spaces $\hat{S}(p,k)$

In (5.5) we introduced the quantity

$$\epsilon_1(p,k) = \inf_{\chi \in \hat{S}(p,k)} \|u_1 - \chi\|_E;$$

clearly

$$\begin{aligned} \epsilon_1(p,k) &= \|u_1 - E(p,k)u_1\|_E \\ &= \frac{\|u_1 - E(p,k)u_1\|_E}{\|u_1\|_E}, \end{aligned}$$

where  $E(p,k)u_1$  is the  $a$ -orthogonal projection of  $u_1$  onto  $\hat{S}(p,k)$  (recall that  $\|u_1\|_E = 1$ ). We have seen in Section 5 that the accuracy of the finite element approximation of the eigenpairs is determined mainly by  $\epsilon_1(p,k)$ . We now present some theoretical and numerical results on the size of  $\epsilon_1(p,k)$ . We first note that (5.2) holds for our choice of  $\hat{S}(p,k)$ . To see this we note that the  $C^\infty$  functions with compact support in  $\Omega$  are dense in  $H_0^1$ . Hence condition (5.2) is equivalent to the approximation of smooth functions by the



p version and the results of [16] lead to the desired conclusion.

From the approximation results in [16] and the regularity results (6.7) - (6.9) we obtain

$$(6.10a) \quad \epsilon_1(p, k) \leq C_1(k) p^{-4/3},$$

$$(6.10b) \quad \epsilon_2(p, k) \leq C_2(k) p^{-8/3},$$

and

$$(6.10c) \quad \epsilon_3(p, k) \leq C_3(k) p^{-1/\epsilon}, \quad \text{for any } \epsilon > 0.$$

These are estimates for the p-version of the finite element method on a fixed mesh; (6.10a) and (6.10b) are optimal, but (6.10c) is not. In fact,  $\epsilon_3(p, k)$  decreases exponentially in  $p$ . The estimates (6.10) (including the refinement of (6.10c) just mentioned) can obviously be written in terms of  $N(p, k)$ . If this is done  $\epsilon_1(p, k)$  and  $\epsilon_2(p, k)$  are seen to decrease algebraically and  $\epsilon_3(p, k)$  exponentially in  $N(p, k)$ .

If  $p \geq 1$  and  $k \geq 0$  are related by

$$\alpha_1(k+1) \leq p \leq \alpha_2(k+1),$$

where  $0 < \alpha_1 < \alpha_2$ , then (see [5] and [14])

$$(6.11) \quad \epsilon_i(p, k) \leq C e^{-\gamma[N(p, k)]^{1/3}}, \quad \text{for some } C > 0 \text{ and } \gamma > 0.$$

$\gamma$  and  $C$ , in general, depend on  $i, \alpha_1, \alpha_2, \beta$  and the domain, but are independent of  $N$ . (6.11), which follows from the fact that  $u_i \in \mathcal{B}_\beta^2(\Omega)$  for every  $i$  (cf. (6.6)), is the basic estimate for the h-p version.

Figure 6.3 depicts the relative error  $\epsilon_1(p, k)$  for various  $k$  as a function of  $p$  in double logarithmic scale. The rate  $p^{-4/3}$  is shown as a slope in the figure (cf. (6.10a)). We see the typical behavior of the p version of the finite element method.

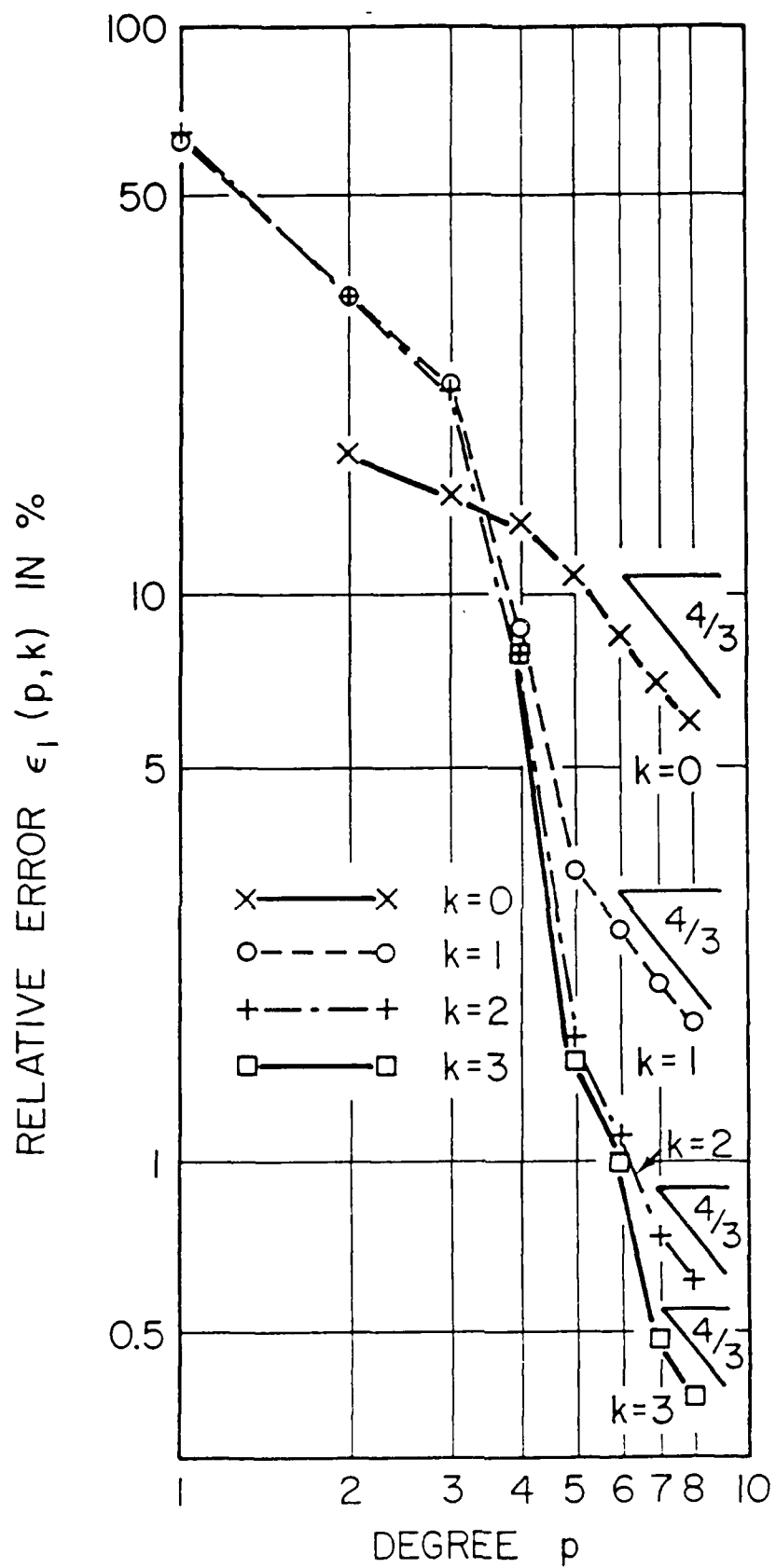


Figure 6 3. The Relative Error  $\epsilon_1(p, k)$ .

For the case  $k = 0$  we see that the rate is very quickly in the asymptotic range  $p^{-4/3}$ . For  $k > 0$  we see the typical reverse S-curve.

In the h-p version we simultaneously increase  $p$  and  $k$ , e.g., consider  $(p,k) = (3,0), (5,1), (7,3), \dots$ . We see that for this sequence, for which  $N(p,k) = 4, 84, 435, \dots$ , the algebraic range typical for the  $p$  version is absent and the error curve is convex with respect to  $N$  (see also Figure 6.7) as follows from (6.11).

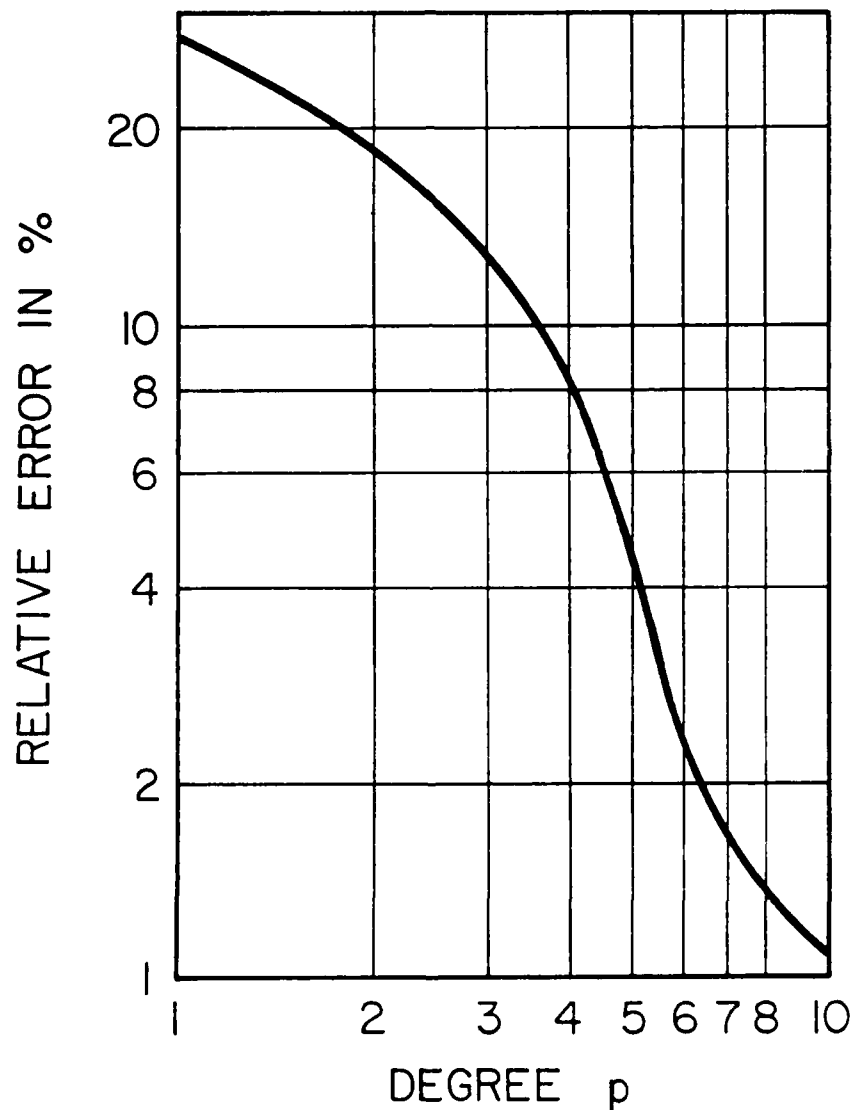


Figure 6.4. The Reverse S-Curve Depicting the Behavior of the  $p$  and h-p Versions of the Finite Element Method

The reverse S-curve shown in Figure 6.4 characterizes the performance of the  $p$  version. It has two parts, the pre-asymptotic phase where the curve turns down and essentially gives the behavior of the  $h$ - $p$  version and the asymptotic phase where the curve is a straight line (with slope  $-4/3$  in our case). As  $k$  increases the asymptotic phase shifts toward higher  $p$  (see Figure 6.3).

To understand this behavior we note that the asymptotic phase begins with that  $p$  at which the error due to the elements with vertices at  $A_1$  (where the leading singularity occurs) starts to dominate the total error. Because  $u_1 \in \mathcal{B}_\beta^2(\Omega)$ , the error decreases exponentially in the pre-asymptotic phase during which it is not dominated by the elements at  $A_1$ . We further note that for  $k$  low (i.e., for an unrefined mesh) the singularity greatly affects the error and thus its influence shows up already for low  $p$ , whereas for  $k$  high (i.e., for a refined mesh) the singularity has been resolved by the mesh, hence has less effect on the total error, and its influence shows up only for high  $p$ . Roughly speaking, the mesh is not properly refined, for the desired accuracy, if we are in the asymptotic phase of the  $p$  version (and the rate of convergence is algebraic). For more details see [2].

Figure 6.5 presents the relative error  $\epsilon_2(p,k)$  for various  $k$  as a function of  $p$ . Here the slope in the asymptotic phase of the  $p$  version is  $-8/3$  (cf. (6.10b)). For  $k = 0$  we are in the asymptotic range for  $p \geq 6$ , but for  $k = 2$  and  $3$  the asymptotic phase is not reached for the values of  $p$  we are considering.

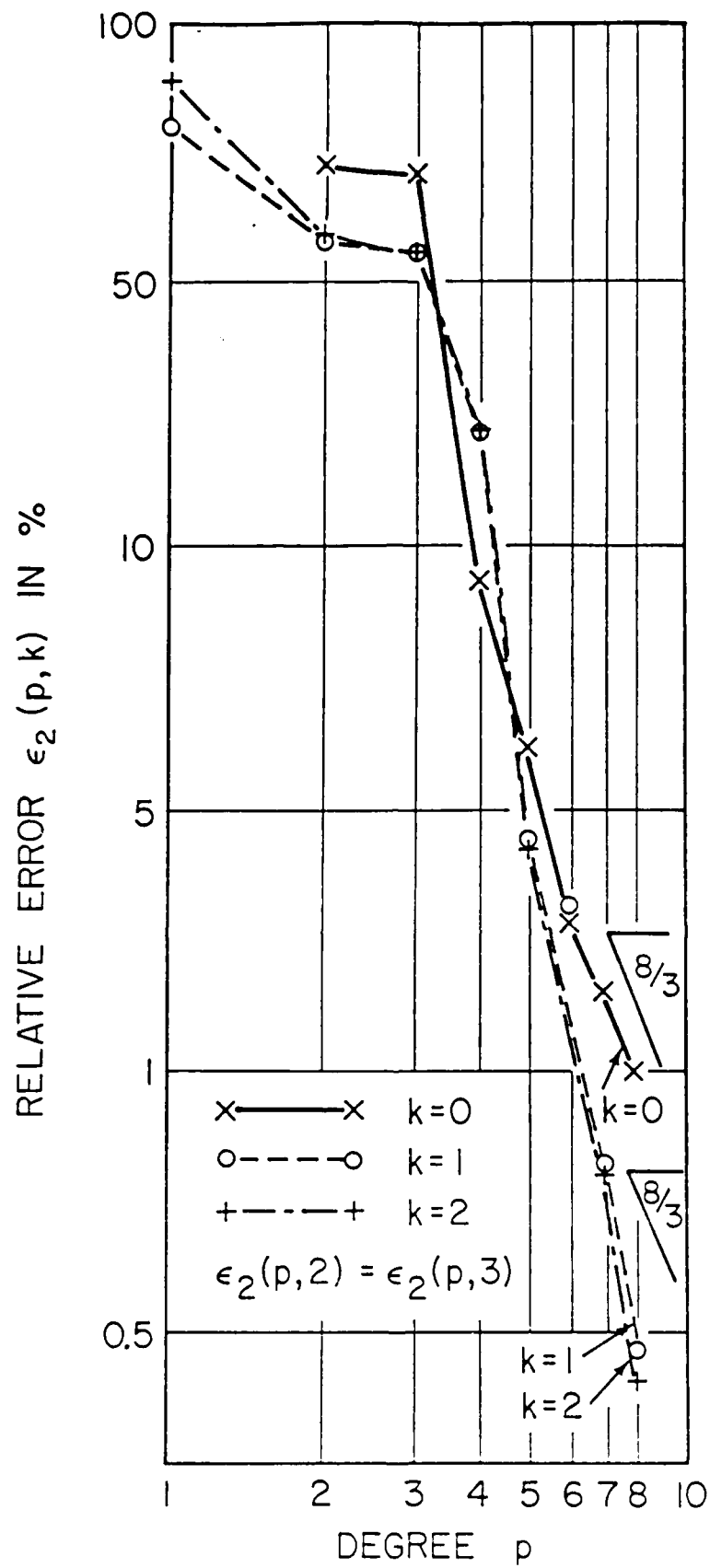


Figure 6.5. The Relative Error  $\epsilon_2(p, k)$ .

Figure 6.3 shows an interesting phenomena occurring in practice, namely, that in certain circumstances refinement can lead to an increase in the error. We note that for fixed  $k$ , the inclusion  $\hat{S}(k, p+1) \supset \hat{S}(k, p)$  is valid and this leads to a decrease in error with increasing  $p$ , while for  $p$  fixed we do not in general have  $\hat{S}(k+1, p) \supset \hat{S}(k, p)$  so that the error is not guaranteed to decrease with increasing  $k$ . For low  $p$ , the major error occurs in the largest elements and in these elements distortion deteriorates approximability in  $H^1(\Omega)$ . A priori, it is virtually impossible to predict this behavior. In general, a conservative strategy for mesh refinement is probably advisable. If one is interested only in very low accuracies, in certain situations an unrefined mesh with undistorted elements could give better results than a refined mesh with distorted elements. Comparing Figures 6.3 and 6.5 we see the interesting feature that for  $p = 2$  the refined mesh gives a better result for the second eigenfunction, while for the first eigenfunctions the unrefined mesh gives the better result. Because we usually compute several eigenvalues with one mesh, we see that the conservative refinement strategy mentioned above is advisable. (We remark that for triangular meshes the situation is different since in this case the spaces are nested.)

Figure 6.6 shows the behavior of  $\epsilon_3(p, k)$ . Since  $u_3$  is analytic, the exponential rate occurs for all  $p$  (cf. (6.10c) and the following discussion) and essentially no difference can be seen in  $\epsilon_3(p, k)$  as a function of  $k \geq 1$ . It is thus best to use an unrefined mesh.

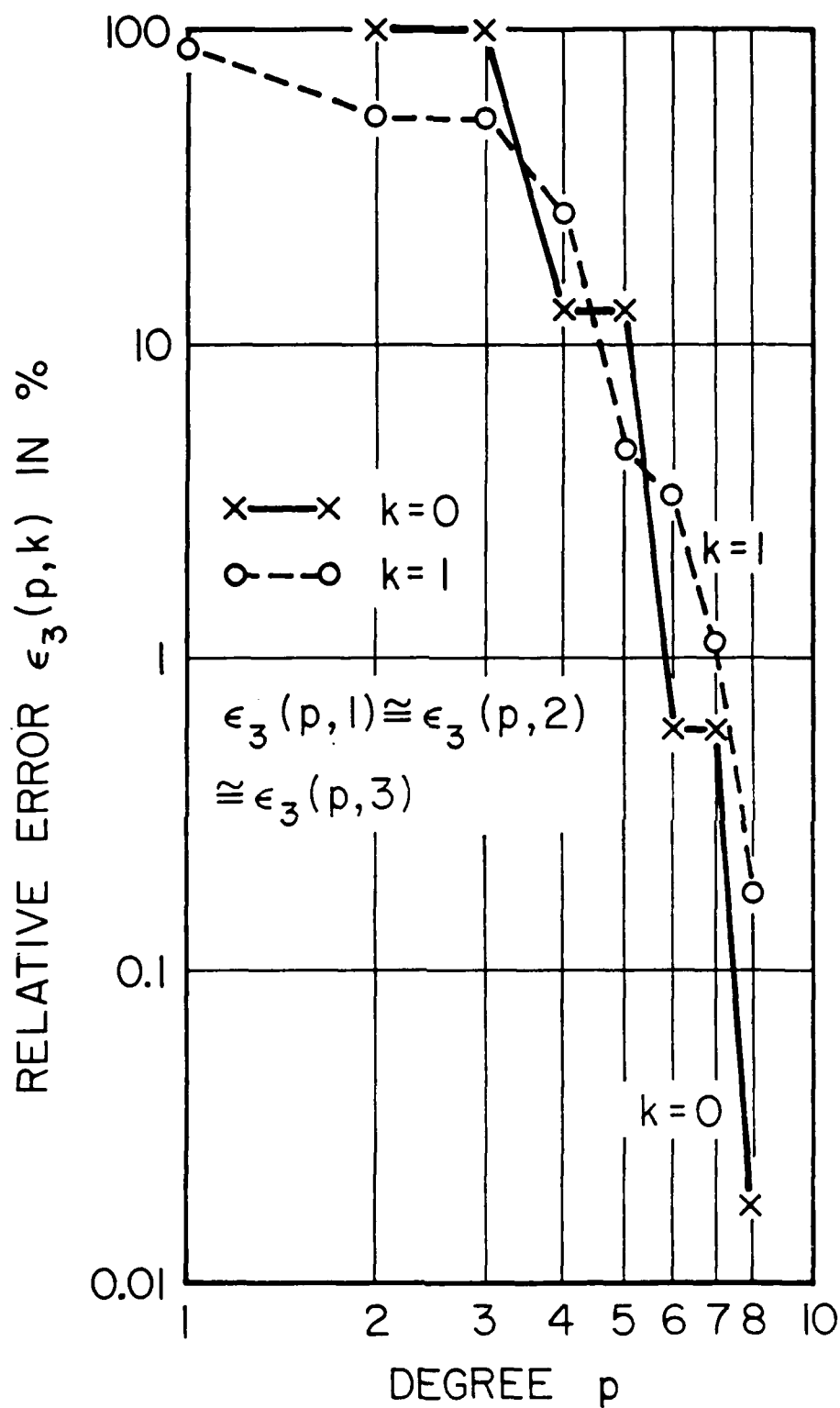


Figure 6.6. The Relative Error  $\epsilon_3(p, k)$ .

Since for  $k = 0$ ,  $u_3$  is symmetric in every element of the mesh, the error for odd  $p+1$  is the same as that for even  $p$ . We note, however, that in general such symmetries are difficult to predict a priori.

Remark 6.1. The numerical results we present require the values of the exact eigenfunctions, which are not explicitly known. We obtained values for these eigenfunctions by a careful extrapolation procedure which we believe leads to reliable results.

#### Accuracy of the Approximate Eigenvalues

Consider the finite dimensional eigenvalue problem (5.3), where the spaces  $\hat{S}(p,k)$  are as described in the Subsection The Finite Element Spaces. With this choice for  $\hat{S}(p,k)$ , (5.3) defines the finite element method for the approximation of the eigenpairs of (6.1). As in Section 5, denote by  $(\lambda_i(p,k), u_i(p,k))$  the eigenpairs of (5.3). These  $(\lambda_i(p,k), u_i(p,k))$  are called the (finite element) approximate eigenpairs. As noted in Section 5,

$$\lambda_i \leq \lambda_i(p,k), \quad \forall p,k, \quad i = 1, \dots, N(p,k) = \dim \hat{S}(p,k),$$

and

$$\lim_{p \rightarrow \infty} \lambda_i(p,k) = \lambda_i, \quad \text{uniformly in } k, \quad \text{for each } i.$$

We are interested here in the accuracy of the approximation  $\lambda_i(p,k) \approx \lambda_i$ ; specifically we are interested in the comparison of  $(\lambda_i(p,k) - \lambda_i)/\lambda_i$  and  $\varepsilon_i^2(p,k)$  (cf. (5.9)). Thus we define

$$(6.12) \quad C_i(p,k) = \frac{(\lambda_i(p,k) - \lambda_i)/\lambda_i}{\varepsilon_i^2(p,k)}.$$

From Theorem 5.1 and the facts that in our case  $\nu(p,k) \leq O(p^{-4/3})$  and  $\eta(p,k) \leq O(p^{-1})$ , uniformly in  $k$ , (cf. [16]), we have

$$1 - O(p^{-2}) \leq C_i(p,k) \leq 1 + O(p^{-4/3}), \quad \text{uniformly in } k.$$



Table 6.2 gives the values of  $C_i(p,k)$  for various values of  $p$  and  $k$ .

	$\begin{matrix} k \\ p \end{matrix}$	0	1	2	3
$\lambda_1$	1	-	1.322	1.349	1.346
	2	0.912	0.988	0.978	0.978
	3	0.967	0.947	0.932	0.932
	4	0.962	0.945	0.929	0.927
	5	0.977	0.991	0.965	0.954
	6	0.984	0.995	0.973	0.962
	7	0.992	0.998	0.997	0.992
	8	0.995	0.999	0.998	0.996
$\lambda_2$	1	-	1.795	1.777	1.777
	2	0.867	0.919	0.921	0.978
	3	0.787	0.892	0.893	0.932
	4	0.780	0.886	0.886	0.927
	5	0.871	0.932	0.931	0.934
	6	0.948	0.932	0.929	0.962
	7	0.966	0.963	0.962	0.991
	8	0.975	0.974	0.975	0.995
$\lambda_3$	1	-	2.692	2.740	2.740
	2	-	1.126	1.128	1.128
	3	1.150	1.121	1.124	1.124
	4	0.831	0.902	0.902	0.901
	5	0.831	0.899	0.899	0.898
	6	0.905	0.904	0.905	0.904
	7	0.905	0.928	0.928	0.928
	8	0.942	0.946	0.946	0.946

Table 6.2. The Values of  $C_i(p,k)$ .

We see that if  $\varepsilon_i(p,k) \leq 5\%$  (cf. Figures 6.3, 6.5, 6.6), then  $C_i(p,k) \approx 1$  and  $C_i(p,k) < 1$ , and furthermore,  $C_i(p,k) \rightarrow 1$  as  $p \rightarrow \infty$ .

The fact that  $C_i(p,k) < 1$  for most values of  $p$  and  $k$  can be

explained in part as follows. Let

$$E(p, k)u = \sum_{j=1}^N a(u, u_j(p, k)) u_j(p, k),$$

be the  $a$ -orthogonal projection onto  $\hat{S}(p, k)$  and let

$$E_i(p, k)u = \sum_{j=i}^N a(u, u_j(p, k)) u_j(p, k).$$

Then we have

$$\begin{aligned} (6.13a) \quad \frac{(\lambda_1(p, k) - \lambda_1)/\lambda_1}{\epsilon_1^2(p, k)} &\leq \left\{ \frac{\|u_1 - E_1(p, k)u_1\|_E^2}{\lambda_1 \|E_1(p, k)u_1\|_b^2} - \frac{\|u_1 - E_1(p, k)u_1\|_b^2}{\|E_1(p, k)u_1\|_b^2} \right\} \frac{1}{\epsilon_1^2(p, k)} \\ &= \left\{ 1 - \lambda_1 \frac{\|u_1 - E_1(p, k)u_1\|_b^2}{\|u_1 - E_1(p, k)u_1\|_E^2} \right\} \times \\ &\quad \frac{\|u_1 - E_1(p, k)u_1\|_E^2}{\lambda_1 \|u_1 - E(p, k)u_1\|_E^2} \frac{1}{\|E_1(p, k)u_1\|_b^2}. \end{aligned}$$

Note that for  $i = 1$ , (6.13a) simplifies to

$$(6.13b) \quad \frac{(\lambda_1(p, k) - \lambda_1)/\lambda_1}{\epsilon_1^2(p, k)} = \left\{ 1 - \lambda_1 \frac{\|u_1 - E(p, k)u_1\|_b^2}{\|u_1 - E(p, k)u_1\|_E^2} \right\} \times \{\lambda_1 \|E(p, k)u_1\|_b^2\}^{-1}.$$

Table 6.3 shows the values of the right side of (6.13b) for  $k = 0$  and  $1$ .

$\begin{array}{c} k \\ p \end{array}$	0	1
1	-	2.04
2	0.916	1.041
3	0.968	0.968
4	0.962	0.952
5	0.977	0.996
6	0.984	0.996
7	0.993	1.000

Table 6.3. The Values of the Right Side of (6.13b) (for  $\lambda_1$ ).

We note two features:

- The values in Table 6.3 are very accurate upper bounds for the corresponding upper bounds in Table 6.2. This indicates that the inequality in (6.13b) is a near equality.
- There are two factors on the right side of (6.13b); the first is less than 1 and the second may be greater than 1. Since the values in Table 6.3 are mostly less than 1 we see that the first factor on the right side of (6.13b) is having a greater influence than the second factor. Thus the cancellation in the first factor is causing the right side of (6.13b) to be less than 1 and

$$\lambda_1 \frac{\|u_1 - E(p, k)u_1\|_b^2}{\|u_1 - E(p, k)u_1\|_E^2}$$

plays the major role in determining the degree to which it is less than 1.

Although we don't have computed values for the right side of (6.13a), Table 6.4 shows the values of

$$(6.14) \quad \left\{ 1 - \lambda_i \frac{\|u_i - E(p, k)u_i\|_b^2}{\|u_i - E(p, k)u_i\|_E^2} \right\} \times \{\lambda_i \|E(p, k)u_i\|_b^2\}^{-1}$$

for  $i = 3$ . We believe this expression is very close to the right side of (6.13a) since  $E_i(p,k)u_i \approx E(p,k)u_i$ . We again see the two features noted above.

$\begin{array}{c} k \\ \backslash \\ p \end{array}$	0	1
1	-	-
2	-	6.52
3	-	1.28
4	0.831	0.932
5	0.831	0.906
6	0.909	0.913
7	0.909	0.941
8	0.948	0.948

Table 6.4. The Values of the Right Side of (6.14) (for  $\lambda_3$ ).

Since  $C_i(p,k) \approx 1$ , the accuracy of the eigenvalue approximation can be read off from Figures 6.3, 6.5, and 6.6, changing the scale of the error by squaring – a 10% approximation error corresponding to a 1% eigenvalue error, a 1% approximation error corresponding to a .01% eigenvalue error, etc.

Figure 6.7 shows the relative error in  $\lambda_1(p,k)$  as a function of the number of degrees of freedom  $N(p,k)$ . The polynomial degree  $p$  is also shown. We see once more the typical S-curve.

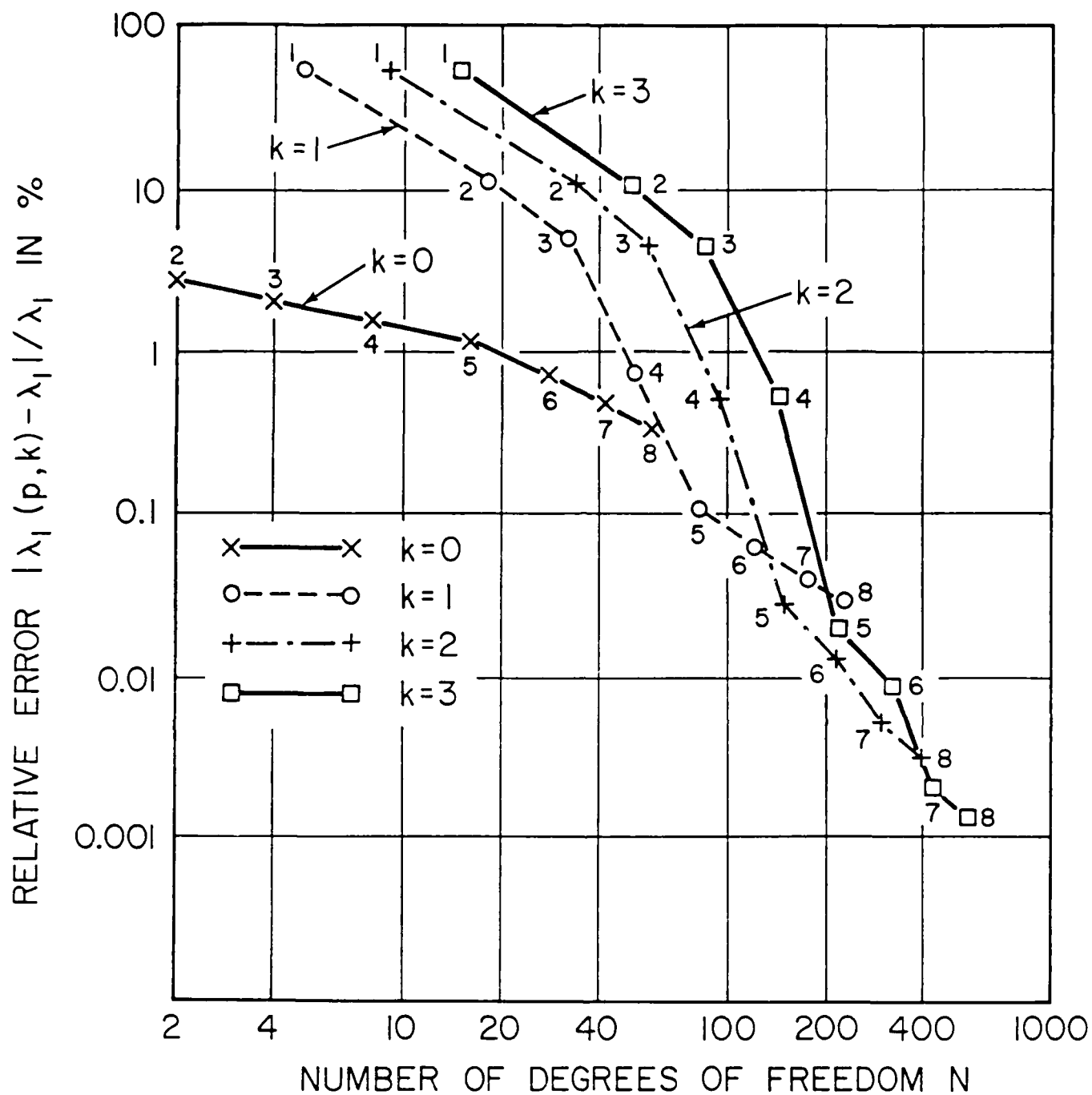


Figure 6.7. Relative Error in  $\lambda_1(p, k)$ .

From Figure 6.7 we note that the smallest eigenvalue  $\lambda_1$  is approximated with an accuracy of  $\leq 3\%$  for  $N = 2$  (corresponding to  $k = 0$  and  $p = 2$ ). From Figures 6.5 and 6.6 and Table 6.1 we see that  $\lambda_2$  and  $\lambda_3$  are approximated

with this accuracy for  $N = 9$  (corresponding to  $k = 0$  and  $p = 4$ ). These results illustrate a feature of eigenvalue approximation: reasonable accuracy for the low eigenvalue can be achieved relatively cheaply.

#### Accuracy of the Approximate Eigenfunctions

Now we turn to a discussion of the accuracy of the finite element approximation of the eigenfunctions. As in the discussion of the accuracy of the eigenvalues we define (cf. (5.10))

$$(6.15) \quad D_i(p, k) = \frac{\|u_i - u_i(p, k)\|_E / \|u_i\|_E}{\epsilon_i(p, k)}.$$

From Theorem 5.1 we have

$$(6.16) \quad 1 \leq D_i(p, k) \leq 1 + O(p^{-4/3}).$$

Table 6.5 presents the values of  $D_i(p, k)$  for various values of  $p$  and  $k$ .

	$\begin{matrix} k \\ p \end{matrix}$	0	1	2	3
$\lambda_1$	1	-	1.060	1.062	1.062
	2	1.004	1.015	1.014	1.014
	3	1.003	1.009	1.008	1.009
	4	1.003	1.003	1.004	1.004
	5	1.002	1.001	1.001	1.001
	6	1.001	1.000	1.001	1.001
	7	1.001	1.000	1.000	1.000
	8	1.000	1.000	1.000	1.000
$\lambda_2$	1	-	1.116	1.115	1.115
	2	1.041	1.038	1.038	1.038
	3	1.049	1.047	1.047	1.047
	4	1.009	1.015	1.015	1.016
	5	1.006	1.001	1.002	1.002
	6	1.001	1.001	1.002	1.002
	7	1.000	1.000	1.000	1.000
	8	1.000	1.000	1.000	1.000
$\lambda_3$	1	-	1.163	1.166	1.166
	2	-	1.039	1.040	1.040
	3	1.414	1.041	1.041	1.041
	4	1.002	1.019	1.019	1.019
	5	1.002	1.004	1.004	1.004
	6	1.001	1.004	1.004	1.004
	7	1.001	1.001	1.001	1.001
	8	1.000	1.000	1.000	1.000

Table 6.5. The Values of  $D_i(p, k)$ .

We clearly see that  $D_i(p, k)$  is very nearly 1 for the entire range of  $p$  and  $k$ . Thus the accuracy of eigenfunction approximation can be read off from Figures 6.3, 6.5, and 6.6. We also see that  $D_i > 1$ ,  $D_i$  converges to 1, but not monotonically, and the convergence appears to be better than  $O(p^{-4/3})$ .

Relation Between the Eigenfunction Error in  $\|\cdot\|_b$  and  $\|\cdot\|_E$

Since  $\eta(p, k) \leq O(p^{-1})$ , (5.11) implies that

$$(6.17) \quad \|u_i - E(p, k)u_i\|_b \leq Cp^{-1} \|u_i - E(p, k)u_i\|_E.$$

Table 6.6 gives the values of

$$(6.18) \quad Q_i(p, k) = \frac{p \|u_i - E(p, k)u_i\|_b / \|u_i\|_b}{\|u_i - E(p, k)u_i\|_E / \|u_i\|_E}.$$

	$\begin{matrix} k \\ p \end{matrix}$	0	1	2	3
$\lambda_1$	1	-	0.688	0.691	0.691
	2	0.745	0.909	0.907	0.907
	3	0.811	1.135	1.153	1.154
	4	1.067	1.016	1.108	1.117
	5	1.055	0.516	0.920	1.047
	6	1.030	0.478	0.956	1.142
	7	0.886	0.520	0.481	0.625
$\lambda_2$	1	-	0.825	0.824	0.824
	2	1.413	1.081	1.081	1.081
	3	1.165	1.578	1.578	1.578
	4	1.877	1.514	1.516	1.516
	5	1.756	1.292	1.299	1.299
	6	1.356	1.553	1.559	1.559
	7	1.289	1.321	1.338	1.338
$\lambda_3$	1	-	0.895	0.897	0.897
	2	-	1.225	1.225	1.225
	3	3.000	1.840	1.839	1.839
	4	1.754	1.774	1.777	1.777
	5	2.193	1.565	1.564	1.564
	6	1.812	1.790	1.791	1.791
	7	2.114	1.693	1.692	1.693

Table 6.6. Values of  $Q_i(p, k)$ .



We see that the values of  $Q_i(p, k)$  are nearly independent of  $p$ , which is what we would expect from (6.17). Note that the first factor on the right side of (6.14) is  $1 - [Q_i(p, k)p^{-1}]^2$ .

Regarding eigenfunction error, (5.12) implies

$$\|u_i - u_i(p, k)\|_b \leq Cp^{-1} \|u_i - u_i(p, k)\|_E.$$

To illustrate this result we define

$$(6.20) \quad \hat{Q}_i(p, k) = \frac{p \|u_i - u_i(p, k)\|_b / \|u_i\|_b}{\|u_i - u_i(p, k)\|_E / \|u_i\|_E}.$$

Table 6.7 presents values of  $\hat{Q}_i(p, k)$  for  $k = 0$ . The results are seen to be similar to those in Table 6.6. We can thus conclude that the errors  $u_i - u_i(p, k)$  and  $u_i - E(p, k)u_i$  have very similar behavior. This is true in either of the norms  $\|\cdot\|_b$  or  $\|\cdot\|_E$  and for rough as well as smooth eigenfunctions, meshes with distorted as well as undistorted elements, and for the entire range of  $p$  and  $k$  that was considered.

$\begin{smallmatrix} \lambda \\ p \end{smallmatrix}$	$\lambda_1$	$\lambda_2$	$\lambda_3$
1	-	-	-
2	0.690	1.193	-
3	0.722	1.934	2.567
4	0.980	1.950	1.709
5	0.964	1.878	2.136
6	0.953	1.382	1.850
7	0.841	1.304	2.158

Table 6.7. Values of  $Q_i(p, 0)$ .

Table 6.8 shows the values of

$$\frac{\|u_i - u_i(p, k)\|_E}{\|u_i\|_E} \quad \text{and} \quad \frac{\|\bar{u}_i - \bar{u}_i(p, k)\|_E}{\|\bar{u}_i\|_E},$$

where  $\|\bar{u}_i\|_b = \|\bar{u}_i(p,k)\|_b = 1$ , for  $k = 0$  and  $1$  and  $i = 1$ . We see (as we should expect) that the first of these two values is the slightly smaller than the second.

p	$\frac{\ u_1 - u_1(p,0)\ _E}{\ u_1\ _E}$	$\frac{\ \bar{u}_1 - \bar{u}_1(p,0)\ _E}{\ \bar{u}_1\ _E}$	$\frac{\ u_1 - u_1(p,1)\ _E}{\ u_1\ _E}$	$\frac{\ \bar{u}_1 - \bar{u}_1(p,1)\ _E}{\ \bar{u}_1\ _E}$
1	-	-	0.666	0.800
2	0.174	0.176	0.339	0.352
3	0.149	0.150	0.235	0.239
4	0.132	0.133	0.088	0.089
5	0.109	0.109	0.032	0.033
6	0.086	0.086	0.026	0.026
7	0.071	0.072	0.020	0.021
8	0.062	0.062	0.017	0.017

Table 6.8. Comparison of the Errors in  $u_1(p,k)$  and  $\bar{u}_1(p,k)$  in  $\|\cdot\|_E$ .

Table 6.9 presents the corresponding errors in  $\|\cdot\|_b$ , i.e.,

$$\frac{\|u_1 - u_1(p,k)\|_b}{\|u_1\|_b} \quad \text{and} \quad \frac{\|\bar{u}_1 - \bar{u}_1(p,k)\|_b}{\|\bar{u}_1\|_b}$$

for  $k = 0$  and  $1$ . The table shows that the second of these quantities is the smaller.

p	$\frac{\ u_1 - u_1(p, 0)\ _b}{\ u_1\ _b}$	$\frac{\ \bar{u}_1 - \bar{u}_1(p, 0)\ _b}{\ \bar{u}_1\ _b}$	$\frac{\ u_1 - u_1(p, 1)\ _b}{\ u_1\ _b}$	$\frac{\ \bar{u}_1 - \bar{u}_1(p, 1)\ _b}{\ \bar{u}_1\ _b}$
1	-	-	0.3175	0.2831
2	0.0601	0.0590	0.1252	0.1175
3	0.0359	0.0345	0.0801	0.0771
4	0.0323	0.0314	0.0029	0.0226
5	0.0210	0.0202	0.0035	0.0034
6	0.0136	0.0131	0.0020	0.0020
7	0.0079	0.0075	0.0005	0.0004
8	0.0068	0.0064	0.0004	0.0004

Table 6.9. Comparison of the Errors  $u_1(p, k)$  and  $\bar{u}_1(p, k)$  in  $\|\cdot\|_b$ .

In general, to measure the error between exact and approximate eigenfunctions we can associate the exact and approximate eigenfunctions together in various ways. For example, we can associate  $u_i$  and  $u_i(p, k)$  or  $\bar{u}_i$  and  $\bar{u}_i(p, k)$ . A third possibility is to associate  $\bar{u}_i(p, k)$  with  $\tilde{u}_i = \beta \bar{u}_i$ , where  $\beta$  is chosen so that  $\|\tilde{u}_i - \bar{u}_i(p, k)\|_E$  is minimal. This choice is of interest since it also minimizes

$$\frac{\|\beta \bar{u}_i - \bar{u}_i(p, k)\|_b}{\|\beta \bar{u}_i - \bar{u}_i(p, k)\|_E}.$$

Table 6.10 shows the values of

$$\frac{\|\tilde{u}_1 - \bar{u}_1(p, k)\|_E}{\|\tilde{u}_1\|_E} \quad \text{and} \quad \frac{\|\tilde{u}_1 - \bar{u}_1(p, k)\|_b}{\|\tilde{u}_1\|_b}.$$

Note that  $\|\tilde{u}_1\|_b < \|\bar{u}_1\|_b$ .

p	$\frac{\ \tilde{u}_1 - \bar{u}_1(p, 0)\ _E}{\ \tilde{u}\ _E}$	$\frac{\ \tilde{u}_1 - \bar{u}_1(p, 0)\ _b}{\ \tilde{u}_1\ _b}$	$\frac{\ \tilde{u}_1 - \bar{u}_1(p, 1)\ _E}{\ \tilde{u}_1\ _E}$	$\frac{\ \tilde{u}_1 - \bar{u}_1(p, 1)\ _b}{\ \tilde{u}\ _b}$
1	-	-	0.808	0.2912
2	0.176	0.0590	0.354	0.1181
3	0.150	0.0345	0.240	0.0773
4	0.133	0.0314	0.089	0.0226
5	0.109	0.0202	0.032	0.0034
6	0.086	0.0131	0.026	0.0020
7	0.071	0.0075	0.021	0.0004
8	0.062	0.0064	0.017	0.0004

Table 6.10. Comparison of the Errors  $\tilde{u}_1 - \bar{u}_1(p, k)$   
in  $\|\cdot\|_E$  and  $\|\cdot\|_b$ .

We see that these different associations of exact and approximate eigenfunctions lead to very similar values for the errors.

#### A Posteriori Estimates

Let  $(\lambda, u) = (\lambda_i, u_i)$  be a fixed eigenpair of (6.1) and let  $(\lambda(p), u(p)) = (\lambda_i(p, k), u_i(p, k))$  be its approximation by the finite element method. We suppress both  $i$  and  $k$  in this notation. Denote by  $z^{[p]} \in H_0^1(\Omega)$  the exact solution of

$$(6.21) \quad \begin{cases} -\Delta z^{[p]} = \lambda(p)u(p) & \text{in } \Omega \\ z^{[p]} = 0 & \text{on } \Gamma, \end{cases}$$

and then denote by  $z_q^{[p]}$  the finite element solution of (6.21) using elements of degree  $q \geq p$  on the same mesh. Obviously  $z_p^{[p]} = u(p)$ , i.e.,  $u(p)$  is the finite element approximation of  $z^{[p]}$ , and  $\|z^{[p]} - z_q^{[p]}\|_E \rightarrow 0$  as  $q \rightarrow \infty$ . Let  $\epsilon^{[p]} = \|z^{[p]} - u(p)\|_E$  and  $\epsilon_q^{[p]} = \|z_q^{[p]} - u(p)\|_E$ .

It is shown in [17] that

$$(6.22) \quad \frac{|\lambda(p) - \lambda|}{\lambda} \leq (\epsilon^{[p]})^2 (1 + o(1)),$$

so that if we have a reliable a posteriori estimate for  $\epsilon^{[p]}$ , we then have a reliable a posteriori estimate for  $\frac{|\lambda(p) - \lambda|}{\lambda}$ . There are a number of ways to obtain an a posteriori estimate for  $\epsilon^{[p]}$ . For a survey of relevant literature we refer to [18]. We now obtain such an estimate. Note that

$$(6.23) \quad \epsilon_q^{[p]} = (\|z_q^{[p]}\|_E^2 - \|u(p)\|_E^2)^{1/2},$$

and for  $q \geq p$ ,

$$\left(\epsilon_q^{[p]}\right)^2 = \|z^{[p]} - u(p)\|_E^2 - \|z^{[p]} - z_q^{[p]}\|_E^2.$$

We thus easily see that  $\epsilon_q^{[p]}$  is increasing with  $q$  and that

$$(6.24) \quad \lim_{q \rightarrow \infty} \left(\epsilon_q^{[p]}\right)^2 = \|z^{[p]} - u(p)\|_E^2 = \left(\epsilon^{[p]}\right)^2.$$

Hence

$$(6.25) \quad \epsilon^{[p]} \approx \epsilon_q^{[p]}, \text{ for } q \text{ large};$$

this is the desired a posteriori estimate for  $\epsilon^{[p]}$ . From (6.22) we have

$$(6.26) \quad \frac{|\lambda(p) - \lambda|}{\lambda} \approx \left(\epsilon_q^{[p]}\right)^2, \text{ for } 1 \ll p \ll q.$$

To check the quality of the approximation in (6.26) we consider the quantity

$$(6.27) \quad Q_q^{[p]} = \frac{(\lambda(p) - \lambda)/\lambda}{\left(\epsilon_q^{[p]}\right)^2}.$$

In Tables 6.11 and 6.12 the values of  $Q_q^{[p]}$  are presented for various values of  $p$  and  $q$  and for 0- and 1-layer meshes for the first eigenvalue.  $Q_\infty^{[p]}$  is obtained by an extrapolation procedure.

$\begin{smallmatrix} p \\ q \end{smallmatrix}$	2	3	4	5	6	7
3	0.167	-	-	-	-	-
4	0.322	0.198	-	-	-	-
5	0.480	0.406	0.256	-	-	-
6	0.6312	0.603	0.506	1.342	-	-
7	0.717	0.717	0.654	1.539	0.306	-
8	0.7659	0.783	0.734	1.662	0.491	0.268
$\infty$	0.843	0.892	0.912	0.925	0.939	0.952

Table 6.11. Values of  $Q_q^{[p]}$  for a 0-layer mesh and for  $i = 1$ .

$\begin{smallmatrix} p \\ q \end{smallmatrix}$	1	2	3	4	5	6	7
2	0.553	-	-	-	-	-	
3	0.701	0.442	-	-	-	-	
4	0.804	0.814	0.778	-	-	-	
5	0.815	0.873	0.881	0.795	-	-	
6	0.816	0.877	0.889	0.855	0.388	-	
7	0.817	0.879	0.893	0.885	0.617	0.379	
8	0.817	0.880	0.895	0.900	0.722	0.552	0.089
$\infty$	0.818	0.881	0.897	0.910	0.920	0.926	0.932

Table 6.12. Values of  $Q_q^{[p]}$  for a 1-layer mesh and for  $i = 1$ .

From the tables we see that  $Q_\infty^{[p]} \approx 1$ , for  $1 \ll p$ , as predicted.

Since  $z_q^{[p]}$  is often cheaper to compute than the approximate eigenvalue  $\lambda(p)$ , we see that (6.26) provides a reasonable estimate for the accuracy of  $\lambda(p)$ ; in fact, for  $q = p+2$  we will usually have

$$|\lambda(p) - \lambda|/\lambda \leq 2 \left( \epsilon_q^{[p]} \right)^2.$$

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